Multidimensional Second-Price and English Auctions*

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January 19, 2015

Job Market Paper

Abstract

In many auctions, because of externalities, each bidder has a different maximum willingness to pay in order to beat each specific competitor, which causes the following new problem. When there are three bidders, two bidders might compete against each other unnecessarily and have worse payoffs than if they had lost to the third bidder, i.e., the two bidders have "group winner regret," which can also lead to inefficiency. While no one-dimensional-bid mechanism is efficient, the Vickrey-Clarke-Groves may require losers to pay. This paper introduces a novel mechanism, the "multidimensional second-price" (MSP) auction (and its open ascending version), and characterizes it. MSP is free of a loser's payment, pairwise stable, and has good incentive properties, including no group winner regret. Moreover, the winner cannot win at any different price by any misreport, and a loser cannot be better off winning by any misreport. MSP is strategyproof for a bidder without externalities imposed by others, and it reduces to the second-price auction when there are no externalities. Simulations suggest that MSP outperforms the second-price auction in terms of both revenue and efficiency.

JEL Classification: C72, C78, D44, D47, D62, L14

Keywords: externalities, core, ex-post regret, collusion, graph, network

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*Please check `http://www.stanford.edu/~ilbello/files/MSP.pdf` for the most up-to-date version.

†I am deeply indebted to my advisors Fuhito Kojima and Peter Glynn, and to my dissertation committee members Jeremy Bulow, Ilya Segal, and Andrei Broder, for invaluable advice, discussions, encouragement, and guidance. I would like to thank Péter Biró, Gabriel Carroll, Itay Fainmesser, Matthew Jackson, Jinwoo Kim, SangMok Lee, Jonathan Levin, Paul Milgrom, Joshua Mollner, Stephen Nei, Michael Ostrovsky, Bobak Pakzad-Hurson, Svetlana Pevnitskaya, Alvin Roth, Shigehiro Serizawa, Fanqi Shi, Andy Skrzypacz, and seminar participants at Stanford for helpful comments and suggestions. This research was supported by the Korea Foundation for Advanced Studies. Correspondence: Department of Management Science and Engineering, Stanford University, 475 Via Ortega, Stanford, CA 94305. Email: swjeong@stanford.edu

1 Introduction

In auctions where the winner imposes identity-dependent externalities on losing bidders, each bidder has a different maximum willingness to pay in order to beat each specific competitor. These externalities exist in many auctions involving commercial bidders. For instance, in a spectrum license auction, the size of externalities can depend on the competitor's coverage map. In the Major League Baseball (MLB) posting auction, externalities can depend on the competitor's league and division (see Appendix A for details).

Even when externalities exist, the auction literature has mainly focused on one-dimensional bids, e.g., a scalar bid in sealed-bid auctions and either "stay" or "drop (out)" (i.e., one button against all competitors in the so-called button model (Milgrom and Weber 1982) in English auctions.[1] However, finding a one-dimensional bidding strategy based on this multidimensional externality information is, in general, difficult.[2] In addition, no one-dimensional mechanism is efficient with respect to the original multidimensional type, and bidders might regret not bidding high or low enough.

Moreover, a group of bidders may have ex-post "group winner regret," which, to the best of my knowledge, has not been studied in the auction literature.[3] Suppose that three companies A, B, and C participate in an English auction for one spectrum license. A and B are two incumbents, competing against each other intensely, but C is assumed to be of no potential threat to either A or B. Then, even if A and B no longer want to compete with C above a certain price, A and B may compete against each other unnecessarily and eventually receive worse payoffs than if they had dropped together before C. However, if only A drops, then B may be better off staying until C drops, and A may regret dropping alone. This regret therefore cannot be solved individually. That is, although A and B competed with each other before, they have incentives to collude by dropping together at a certain price, as "yesterday's enemy is today's friend."

Although collusion is normally prohibited in many auctions, the incentive to collude may still exist to avoid group winner regret. Furthermore, group winner regret-freeness may

[1] Jehiel, Moldovanu, and Stacchetti (1996) is a pioneering paper dealing with externalities in auctions. See also Jehiel and Moldovanu (1996), Caillaud and Jehiel (1998), Jehiel et al. (1999), Jehiel and Moldovanu (1999), Segal (1999), Jehiel and Moldovanu (2000), Varma (2002), Rhee (2007), Aseff and Chade (2008), Jeziorski and Segal (2009), Rhee (2010), and Hu et al. (2013). Jehiel et al. (1996) show a multidimensional optimal auction that is not free of a loser's payment. Jehiel et al. (1999) show the conditions of incentive compatible and individually rational multidimensional bid auctions, but they do not present any practical multidimensional auction and then focus on one-dimensional auctions.

[2] English auctions with externalities lack closed-form solutions in general (De Castro and Karney 2012).

[3] Although the following motivating example in the main text may seem intuitive, the formal definition of group winner regret is nontrivial since we cannot say A and B regret not losing to C unless the winning of C can be "justified" in some sense. The definition of ex-post group winner regret is one of the crucial parts of this paper. All regret considered in this paper is ex-post; thus, "ex-post" will be normally omitted.

increase efficiency.[4] Thus, it is desirable to make a mechanism free of group winner regret.

I present a novel multidimensional mechanism, the "multidimensional second-price" (MSP) auction that is free of group winner regret. MSP is pairwise stable, i.e., one bidder and the auctioneer cannot block the outcome; thus, it does not suffer the "low revenue problem." MSP has good incentive properties, including no group winner regret. Additionally, the winner cannot win at any different price by any misreport, and a losing bidder cannot be better off winning by any misreport. Moreover, MSP is strategyproof for a bidder without externalities imposed by others. When there are no externalities for all bidders, MSP (and its open ascending version) reduces to the standard one-dimensional second-price (and English) auction mechanism.

Therefore, the new mechanisms not only have good properties, but can also be seen as natural multidimensional extensions of the second-price and the English auctions, which makes them easily understandable by and acceptable to both bidders and the auctioneer. Extending the standard (one-dimensional) second-price and English auctions into multiple dimensions may appear simple. However, comparing bids pairwise to determine the winner can induce a cycle, e.g., bidder 1 beats bidder 2, and bidder 2 beats bidder 3, but bidder 3 beats bidder 1. Therefore, it is nontrivial to define the winner and the second-price. I also characterize MSP.

Externalities often tend to be inadvertently ignored so that the second-price auction (one-dimensional VCG) seems to be efficient, incentive compatible, and individually rational, which is not true with respect to the actual multidimensional type. In fact, simulations suggest that MSP outperforms the second-price auction in terms of both revenue and efficiency. The intuition behind this is that group winner regret-freeness tends to increase efficiency, and pairwise stability tends to increase both revenue and efficiency.

The remainder of this introduction explains the problems of existing mechanisms and the importance of the properties that MSP satisfies in detail. An outcome of the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey 1961; Clarke 1971; Groves 1973) may not be in the core, as it may not be in package auctions (Ausubel and Milgrom 2006). Other problems with VCG arise in settings with externalities, as discussed in detail in Jeong (2014). First, it may require losing bidders to pay. This paper considers "loser's payment"-free (LPF) mechanisms. Thus, another candidate, the menu auction (Bernheim and Whinston 1986), which is not LPF, is also excluded. Even when externalities exist, LPF auctions (e.g., second-price and English) are widely used. Although I do not argue that a loser's payment is undesirable, I present

[4] The intuition is that if many bidders prefer another allocation, then it is likely that the current allocation is inefficient. In some models, including the example in the previous paragraph, group winner regret implies inefficiency, but this is not true in general. However, simulations in Section 6 support this intuition.

some possible reasons why some markets might not prefer a loser's payment.[5] Of course, losing bidders may want to pay when there exist huge negative externalities, such as nuclear weapons (Jehiel et al. 1996). However, externalities are normally more difficult to estimate than the own valuation because they can change in response to the future environment. For instance, a losing bidder may want to improve the relationship with the winner to decrease the externality. In addition, the winner might have the "winner's curse," or might not use the item for a long time, in which case it might be less threatening than expected. That is, a loser's payment can be seen as an immediate payment for an uncertain future. Thus, losing bidders may not want to pay, and enforcing a loser's payment can discourage participation. If participation decreases, revenue can decrease (Bulow and Klemperer 1996). Second, the low revenue problem is more serious. VCG may result in a negative payment (subsidy) as Myerson and Satterthwaite (1983) indicate. Also, a reserve price cannot be simply implemented by treating the auctioneer as a bidder or by introducing a minimum bid. Third, the "shill bidding problem" (Yokoo et al. 2004) is more serious: there exists a weakly dominant shill bidding strategy, and VCG may result in an undefined outcome.

When VCG fails to produce a core outcome, one possible solution might be a core-selecting mechanism, as in core-selecting package auctions (Day and Milgrom 2008; Day and Raghavan 2007; Day and Cramton 2012; Erdil and Klemperer 2010; Parkes et al. 2001). However, Jeong (2014) shows that there is no loser's payment-free core-selecting mechanism. In fact, the core property might be less important than it is in package auctions. In package auctions, if an outcome is not in the core, each bidder in a blocking coalition is willing to pay to receive some item. In contrast, when externalities exist, some bidders need to pay not to receive the item, but to decrease negative externalities by changing the allocation. Thus, for the same reasons that a loser's payment might be undesirable, bidders in the blocking coalition might be willing to accept the non-core outcome.[6]

However, a mechanism still needs to have a certain degree of stability and "fairness." Then, a well-known alternative to the core is pairwise stability (PS), where a pair consists of one bidder and the auctioneer. If an outcome is not pairwise stable, then the winning price is less than some losing bidder's bid against the winner, which might be too disputable or easily deviated by one bidder and the auctioneer. Thus, I impose loser's payment-freeness (LPF) and, as the second requirement, PS. Then, unfortunately, neither incentive compatibility

[5]For efficiency (and a certain degree of incentive to participate, e.g., weak IR in this paper), a loser's payment is necessary. Note that, however, if we allow resales (but without commitment) efficiency is impossible even with a loser's payment, as shown in Jehiel and Moldovanu (1999). See also footnote 6.

[6]One possible way to indirectly see the desirability of a loser's payment or the core property is a hybrid approach. For instance, let the auctioneer run MSP first as a default outcome and then run a core-selecting mechanism with the same bids. If the outcomes of two auctions are different, then let bidders in a blocking coalition have a chance to block the MSP outcome, which normally requires a loser's payment.

(IC) nor individual rationality (IR) is possible even with inefficient mechanisms.

A mechanism should nevertheless have a certain degree of incentive to participate for each individual since we cannot simply make the winner arbitrarily overpay to achieve other desirable properties. In fact, IR might be too strong a requirement, especially when it is difficult to predict the outcome of nonparticipation. Weak IR, i.e., the payoff of participation is at least as large as the worst payoff of nonparticipation, might be sufficient and desirable since it is what the bidders are guaranteed to receive with nonparticipation. An outcome that is not weakly IR implies that the winner needs to pay more than the maximum bid against all bidders, which may be undesirable and discourage participation. For instance, adjusting the payment of the winner to try to make the outcome in the core can fail to meet weak IR. In contrast, MSP is weakly IR. IR implies weak IR, and weak IR is also a well-known alternative to IR in the core with externalities literature.

Unfortunately, however, even weak IR cannot be attained with efficiency. Although efficiency (of bidders and the auctioneer) is normally desired in the literature, there may exist an inefficient but pairwise stable outcome that has lower revenue than a minimum-revenue core outcome and higher bidder surplus. That is, efficiency (and the core property) may come from the sacrifice of bidders if they are willing to accept pairwise stable outcomes. Thus, if we consider the welfare of bidders only (or put more weight on this), or consider other surplus (e.g., consumer's), efficiency might be less important.

Whereas individual rationality is easy to achieve when there are no externalities, incentive compatibility is often impossible with other desirable properties. However, a mechanism should have good incentive properties. For instance, minimum-revenue core-selecting package auctions minimize the sum of incentives to deviate, and the generalized second-price auction (Edelman et al. 2007; Varian 2007) is "locally envy-free," i.e., each bidder does not envy another bidder who is either right above or right below her. Likewise, MSP is free of certain regrets that can be considered good incentive properties, and it is strategyproof for a bidder without externalities imposed by others.

The remainder of this paper is organized as follows. Section 2 illustrates group winner regret and the new mechanisms with the motivating example. Section 3 introduces the model and the impossibility results regarding mechanisms with LPF and PS. Section 4 introduces ex-post regret, a bid network graph, and a mechanism equivalent to MSP. Section 5 introduces the new mechanisms, MSP and ME (Multidimensional English). Section 6 shows the simulation results. Section 7 concludes the paper. Appendix A discusses an application, the MLB posting system. Appendix B discusses the core with externalities. Appendix C presents additional characterization results. All proofs can be found in Appendix D.

2 Illustrative motivating example

I will informally illustrate group winner regret and the new mechanism, MSP, with the motivating example mentioned in the introduction. First, I use the English auction to show group winner regret and then describe MSP, which is free of group winner regret.[7]

The auctioneer sells one item to three bidders without a reserve price. The type profile of bidders is $T = \begin{bmatrix} 16 & -6 & 0 \\ -12 & 20 & 0 \\ 0 & 0 & 21 \end{bmatrix}$ (or $B = \begin{bmatrix} 0 & 26 & 21 \\ 28 & 0 & 21 \\ 16 & 20 & 0 \end{bmatrix}$), where the j-th column vector is the type of bidder j; e.g., bidder 1's valuation on the item is 16, and bidder 1 suffers negative externality of 12 if bidder 2 wins. Then, j's maximum willingness to pay in order to beat i is $b_{ij} \equiv t_{jj} - t_{ij}$; e.g., bidder 1 is willing to pay up to $28 = 16 - (-12)$ to beat bidder 2. Note that this is the situation in the motivating example, i.e., bidders 1 and 2 are willing to pay more in order to beat each other than to beat bidder 3.

In the English auction, for simplicity, let the bidding function of bidder j be

$$\beta^I \left(\mathbf{b}_j, R \right) \equiv \max_{i \in R \setminus \{j\}} \left\{ b_{ij} \right\}, \tag{2.1}$$

where R is the set of remaining bidders at the current price, i.e., each bidder stays until the maximum bid against all remaining competitors.[8] As mentioned earlier, the English auction with β^I can be implemented as a direct mechanism, which is φ^I in Algorithm 1.

One reason why a *bid matrix*, B, is useful (compared to a type matrix, T) is that we can interpret an English auction as updating a *bid graph*, a graph representation of a bid matrix. In a bid graph, each node represents a bidder, and each edge represents a bid. Figure 2.1 shows the (q,k)-*undominated bid graph*,[9] which will be simply referred to as (q,k)-graph or (q,k), where q is the current price, and k is the step (starts from 0) at the same price q. Note that in English auctions, one bidder's drop can lead to another bidder's drop at the same price; thus, k is used to distinguish each step at the same price. For illustrative purposes, in each $(q,0)$-graph, the edge with price q, which will be removed in the next $(q,1)$-graph, is shown with a dashed arrow (colored in red).

The English auction starts from price $q = 0$. At $q = 16$, bidder 1 no longer needs to

[7] I define group winner regret with direct mechanisms, and MSP is a direct mechanism. For an intuitive explanation, however, I use a dynamic auction, the English auction, to explain group winner regret because what *drop* and *remaining bidders* mean are clear. Note that a dynamic mechanism with some strategies can be implemented as a direct mechanism that runs a dynamic auction "internally" after receiving bids.

[8] This is not an equilibrium. However, there is no closed-form equilibrium in English auctions in general. Even in a simple model (e.g., Varma 1999) where an equilibrium (which is unnecessarily complicated for an intuitive explanation) exists, group winner regret can occur. Thus, for simplicity, β^I is used.

[9] Here, details of (q,k)-(group) undominated bid graph in Figures 2.1 and 2.2 are unnecessary.

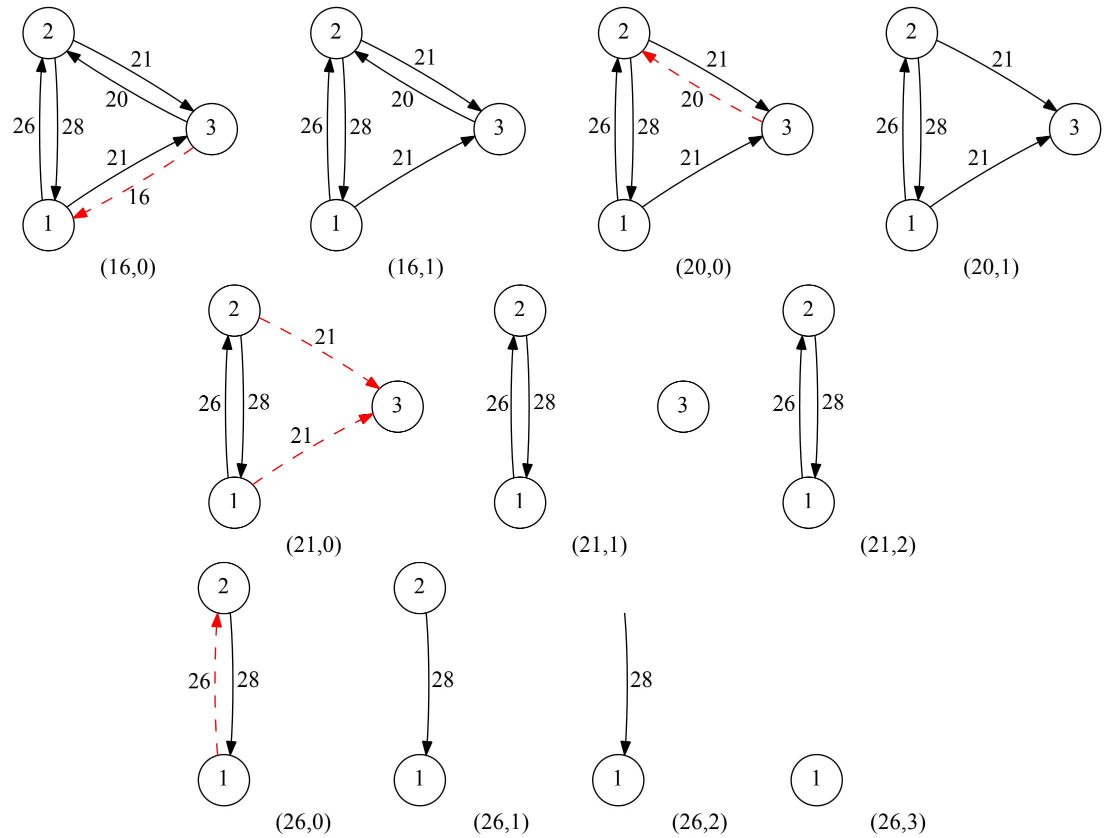

Figure 2.1: (q, k)-undominated bid graph of the motivating example

bid against bidder 3, but she still needs to stay due to bidder 2, which is shown in $(16, 1)$. Likewise, at $q = 20$, bidder 2 no longer needs to bid against bidder 3, but she still needs to stay due to bidder 1. At $q = 21$, bidder 3 no longer needs to bid against either bidder 1 or 2, i.e., bidder 3's staying any longer is dominated. Thus, bidder 3 drops. Then, only two bidders are left, which in fact makes β^I a dominant strategy for bidders 1 and 2. Thus, bidder 2 drops at $q = 26$, and bidder 1, the only remaining bidder, wins at $p = 26$. Then, the payoffs of the bidders are $\mathbf{u} = (-10, -6, 0)$. However, the payoffs when bidder 3 wins at some price p are $\mathbf{u} = (0, 0, 21 - p)$. Thus, bidders 1 and 2 are better off losing to bidder 3, and they could have lost to bidder 3 if they had dropped together earlier than bidder 3. That is, bidders 1 and 2 have group winner regret. Moreover, the outcome is inefficient.

Roughly, group winner regret consists of the following two conditions. First, a group of bidders are better off losing to any of the "remaining" bidders. Second, the group could have lost to some remaining bidder. The second condition "justifies" the regret, i.e., even if they are better off losing to some bidder, they cannot say they "regret" if there is no way for them to lose to the bidder. The first condition not only enables some mechanism (e.g., MSP) to be free of the regret, but also makes the group regret "greater" in the following sense. Suppose the group regrets not losing to *one specific remaining bidder*, as opposed to *any of the remaining bidders*. Then, it is impossible to make a mechanism free of this regret. Moreover, even if there is such regret, the group might regret "less" because it is difficult for bidders to forecast who will win. However, they might regret "greater" if they could have been better off no matter which remaining bidder had won.

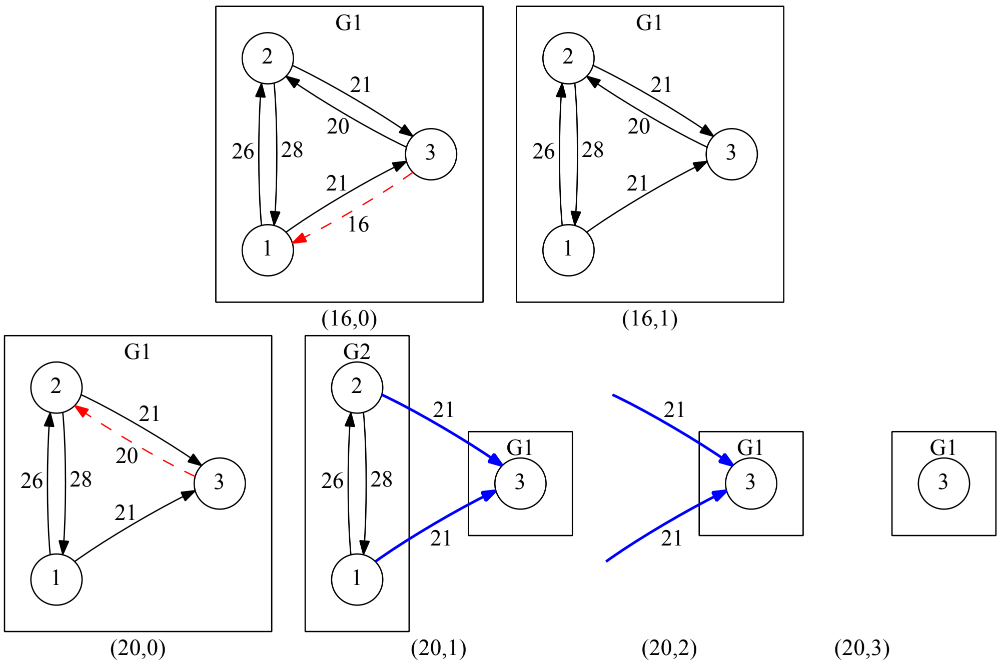

Figure 2.2: (q, k)-group undominated bid graph of the motivating example

I will now show that the two conditions are satisfied in the above example. Figure 2.2 shows the (q, k)-*group undominated bid graph*, which now shows "group." A thicker arrow (colored in blue) denotes a bid across groups. Until $(20, 0)$, there is only one group, but at $(20, 1)$, there are two groups, G1 and G2. G1 has bidder 3, and G2 has bidders 1 and 2. The two members in G2 bid only against each other, not against G1, while G1 still bids against G2. This is "unnecessary" internal competition in G2 because they are better off losing to any bidder in G1 (in this example, bidder 3, since it is the only bidder) at any price $p > 20$. That is, their staying above 20 is dominated by dropping *together* at 20. In MSP, bidder 3 wins at 20. This is the main idea behind group winner regret and MSP. MSP finds groups whose members are in unnecessary internal competition and drops such groups.

A group and its unnecessary internal competition can be intuitively recognized by the connectedness of a bid graph. A *strong component* (or *strongly connected component*) is a maximal subgraph that has a bidirectional path between each pair of nodes. A strong component can then be interpreted as a group. In general, there can be more than two groups, and any group can have more than one bidder.

MSP (in fact, φ^{II}, since MSP is a simplified version of φ^{II}, as explained below) is a "groupwise" version of φ^{I}. That is, φ^{II} is a direct mechanism version of the English auction with the following bidding function for any bidder in group S,

$$\beta^{II}(\mathbf{b}_S, R) \equiv \max_{i\in R\setminus S, j\in S}\{b_{ij}\}, \tag{2.2}$$

where R is the set of remaining bidders at the current price, i.e., each bidder in a group stays until the maximum bid against all remaining competitors outside the group.

Roughly, φ^{II} works as follows (see Example 3 for more examples). Each bidder j submits $\mathbf{b}_j$. φ^{II} increases price q to find a bid $b_{cm} = q$ that has not been removed (or *unblocked*). Then, φ^{II} unblocks b_{cm}, which implies that m no longer needs to bid against c above price q. If the unblocking still results in only one group, then φ^{II} finds the next unblocking. Otherwise, i.e., if multiple groups exist, then φ^{II} finds the group that does not bid against any other groups and then drops all bidders in the group. If bidder j drops, then the bidders who bid against j no longer need to bid against j, thus φ^{II} unblocks those bids, which will be called a *sequential unblock,* i.e., one unblocking *sequentially* leads to another unblocking at the same price. If multiple groups still exist, this leads to some other group's drop, which will be called a *sequential drop.* This may, in turn, lead to another sequential unblock, and so on. That is, sequential unblock and drop can be repeated at the same price.

Interestingly, when there are multiple groups, the "path" (called here the *externality chain*) of groups is always well-defined, i.e., it always starts at the group including m (who just gave up bidding) and ends at the group including c (whom m just gave up bidding against), even when there are multiple paths, as in Example 3-(3) (Figure 5.2). This enables the simplification of φ^{II} into MSP. That is, MSP can simply drop (all the bidders in) all groups except for the group including c since no matter how many groups and externality chains exist, any externality chain always ends at the group including c. This process is repeated until one bidder, the winner (the last c), is left. Only the winner pays, and the price is the last $q = b_{cm}$, the last bid that is unblocked, which is defined as the *second-price* in MSP. By its nature, MSP can be easily implemented as a dynamic open ascending auction, the multidimensional English (ME in Algorithm 4) auction mechanism.

I have roughly explained all four mechanisms, φ^{I}, φ^{II}, MSP, and ME (Algorithms 1-4), of this paper. They can be easily understood since they are closely related: φ^{I} is a direct mechanism version of the English auction with β^{I}; φ^{II} is a "groupwise" version of φ^{I}, i.e., a direct mechanism version of the English auction with β^{II}; MSP is a simplified but equivalent version of φ^{II}; and ME is a dynamic open ascending version of MSP. In particular, φ^{I} and φ^{II} will be used to define winner's regret and group winner regret, respectively.

3 Multidimensional auctions with externalities

3.1 Model

There are one indivisible item, one auctioneer (denoted by 0), and a set of bidders $N = \{1, 2, ..., n\}$. $N^0 = N \cup \{0\}$. Each bidder only knows her own valuation of the item and the externalities imposed by others. That is, the type of bidder j is denoted by a column vector $\mathbf{t}_j = (t_{ij})_{i \in N^0}$, $t_{ij} \in \mathbb{R}$, where t_{ij} denotes the externality ($t_{ij} < 0$: negative externality, $t_{ij} > 0$: positive externality) imposed on bidder j when bidder $i \neq j$ wins the item, and t_{jj} denotes the bidder j's own valuation of the item. The independent private value is assumed, i.e., each t_{ij} is independent, and knowing another's type does not affect her own valuation. The profile of types of n bidders and the auctioneer is denoted by a $(n+1)$-by-$(n+1)$ ***type matrix***, $T = (t_{ij})_{i,j \in N^0}$. The utility of bidder j is denoted by u_j^T, and $u_j^T = t_{wj} - p_j$ when her payment is p_j and bidder w wins.

For a given $\mathbf{t}_j$, $b_{ij} \equiv t_{jj} - t_{ij}$ is bidder j's maximum willingness to pay in order to beat bidder i, which will be called "*j's* ***bid against*** *i*." Let $B = (b_{ij})_{i,j \in N^0}$ be called a ***bid matrix***. A column vector $\mathbf{b}_j$ can also be interpreted as the normalization such that the valuation is zero, $b_{jj} = 0$, and each b_{ij} is the externality ($b_{ij} > 0$: negative externality, $b_{ij} < 0$: positive externality). The normalized utility of bidder j is denoted by u_j^B, and $u_j^B = -b_{wj} - p_j$ when bidder w wins. For simplicity, u_j will be used instead of u_j^T (unnormalized) or u_j^B (normalized) when the reference type is unambiguous. Unless otherwise specified, $t_{j0} = 0$ and $t_{0j} = 0$ for all $j \in N^0$, i.e., the auctioneer neither suffers nor imposes externalities. Then, T induces a unique B. Thus, without loss of generality, we can consider direct mechanisms in which each bidder j submit its bid $\mathbf{b}_j$ instead of its type $\mathbf{t}_j$.[10]

Nonparticipation or drop (in dynamic auctions) of some bidders does not affect the remaining bidders' types, i.e., even if only a subset of bidders $S \subseteq N^0$ participates or remains in an auction, t_{ij} for $i, j \in S$ does not change. Each bidder's outside option is determined by the outcome of an auction, i.e., even if bidder j does not participate in an auction, $u_j = t_{ij}$ if bidder i wins. The auctioneer can neither extract payments from nonparticipants, nor give them the item. Let T_S denote the $|S|$-by-$|S|$ submatrix of T that has only rows and columns $j \in S$. Likewise, $T_{-S} \equiv T_{S^C}$, where $S^C = N^0 \backslash S$. Abusing the notation, $\{j\}$ can be written as j. Note that T_j is 1-by-1, and T_{-j} is n-by-n, whereas $\mathbf{t}_j$ is $(n+1)$-by-1, and $\mathbf{t}_{-j}$ is $(n+1)$-by-n.

[10] A bid matrix is notationally simpler and easier for utility comparisons when there is no loser's payment, e.g., if j wins at price p, then j is better off losing to i if "$p > b_{ij}$," which is equivalent to "$t_{jj} - p < t_{ij}$" using a type matrix. However, a type matrix is the usual way to describe externalities in the literature. Thus, T is used for preliminary discussions and VCG, and from Section 4 on, B is used.

Each $b_{ij} \in \left[\underline{b}, \bar{b}\right]$, where $\underline{b}, \bar{b} \in \mathbb{R}$ are fixed bounds.[11] For a given subset of bidders $S \subseteq N^0$ and $j \in S$, let $\mathcal{B}_{j,S}$ be the set of all possible $\mathbf{b}_j = (b_{ij})_{i \in S}$ that satisfy this constraint, i.e., $\mathcal{B}_{j,S} = \left\{(b_{ij})_{i \in S} \in \left[\underline{b}, \bar{b}\right]^{|S|}\right\}$.[12] Let $\mathcal{B}_S = \prod_{j \in S} \mathcal{B}_{j,S}$ and $\mathcal{B} = \mathcal{B}_{N^0}$. Likewise, $\mathcal{T}_{j,S} = \left\{(t_{ij})_{i \in S} \in \mathbb{R}^{|S|} \middle| (b_{ij})_{i \in S} \in \left[\underline{b}, \bar{b}\right]^{|S|}\right\}$, $\mathcal{T}_S = \prod_{j \in S} \mathcal{T}_{j,S}$, and $\mathcal{T} = \mathcal{T}_{N^0}$. In particular, $\underline{\mathbf{b}} = (\underline{b}, \underline{b}, ..., \underline{b})^T$ denotes the *lowest bid*, and $\mathbf{t}_j$ is said to be the *lowest type* if $\mathbf{b}_j = \underline{\mathbf{b}}$.

For simplicity, normally only the types of bidders, T_N, will be given in examples when $t_{j0} = 0$ and $t_{0j} = 0$ for all $j \in N^0$. Note that t_{00} can be interpreted as the valuation or the reserve price of the auctioneer, although not all mechanisms can implement a reserve price in this way (see Example 3-(5)).

A direct auction mechanism is a pair of functions $\varphi = (x, \rho)$, $\varphi : \mathcal{T} \to (N^0, \mathbb{R}^{n+1})$,[13] where x is the winner-determination rule and ρ is the payment rule. $w \equiv x(T)$ is the winner (can be the auctioneer, i.e., no-sale), and $\mathbf{p} \equiv \rho(T)$ is the payment of all bidders and the auctioneer, and $p \equiv \sum_{j \in N^0} p_j$ is the sum of payments. Let $X : \mathcal{T} \to 2^{N^0}$ be the set-valued map such that $W \equiv X(T)$ is the set of all possible winners due to ties.

A mechanism φ is said to be *loser's payment-free* (LPF)[14] if $\rho_j(T) = 0$ for all $j \neq w$ and $T \in \mathcal{T}$. An auction outcome is said to be *efficient* if $w \in \arg\max_{i \in N^0} \sum_{j \in N^0} t_{ij}$, i.e., it only considers bidders and the auctioneer, which is standard in auction theory. A mechanism φ is said to be efficient if the outcome is efficient for all $T \in \mathcal{T}$.

When $S^C = N^0 \backslash S$, $0 \in S$, refuses to participate, φ runs an auction with subtype T_S, where φ is implicitly defined in the subspace, $T = T_S$ and $\mathcal{T} = \mathcal{T}_S$.[15] Therefore, for $S \subseteq N^0$, the *coalition value function with respect to* $\varphi = (x, \rho)$ is defined as[16]

$$v(S) = v(S; T, \varphi) = \begin{cases} \max_{i \in S} \sum_{j \in S} t_{ij} & 0 \in S \\ \sum_{j \in S} t_{x(T_{-S}),j} & 0 \notin S \end{cases}.$$

[11] In general, $\underline{b}$ can be negative, and t_{jj} itself can be negative as well, e.g., auctioning off garbage, which might have a negative value for many bidders, but positive for recycling companies.

[12] In $\mathcal{B}_{j,S}$, S is needed when only a subset of bidders S participates, i.e., j reports the subtype, $(t_{ij})_{i \in S}$. Abusing the notation, $\mathcal{B}_j$ can be used instead of $\mathcal{B}_{j,S}$ when S is unambiguous. Likewise, $\mathcal{B}$ and B can be used instead of $\mathcal{B}_S$ and B_S. The same rule applies to $\mathcal{T}_{j,S}$, $\mathcal{T}_S$, and T.

[13] $\varphi : \mathcal{B} \to \left(N^0, \mathbb{R}^{n+1}\right)$ when B is used instead of T.

[14] "Loser's payment-free" and "free of a loser's payment" are used interchangeably here. Likewise, "no loser's payment" and "loser's payment-freeness" are used interchangeably. The suffix form, "-free," is introduced to use the same acronym for both noun and adjective forms, e.g., LPF can stand for both "loser's payment-free" and "loser's payment-freeness." The same rule applies to any regret, i.e., "XYZ regret-free" and "free of XYZ regret"; "XYZ regret-freeness" and "no XYZ regret."

[15] When no bidders participate, i.e., $S = \{0\}$, the auctioneer keeps the item, and $\mathbf{p} = \mathbf{0}$. Abusing the notation, even when only S participates, $\mathbf{p} \in R^{N^0}$.

[16] When a tie exists in $x(T_{-S})$, the chosen value is maintained for $v(S)$. Abusing the notation, "$v(S) = v(S; T, \varphi)$" means that omitted T and φ are clear from the context when $v(S)$ is used.

Remark. When externalities exist, there can be alternative coalition value functions depending on which strategies the outside players play, which is a concept of *effectiveness*. See Appendix B for details. However, the coalition value function and the core are only used for the impossibility result, and the impossibility holds with alternatives.

For a given payoff profile $\mathbf{u}$, if $\sum_{j\in S} u_j < v(S)$ for some $S \subseteq N^0$, then S is called a *blocking coalition*, and $\mathbf{u}$ is said to be *blocked* by S. The *core* is defined as the set of payoff profiles that are feasible, i.e., $\sum_{j\in N^0} u_j = v(N^0)$, and not blocked by any coalition. That is,

$$Core\left(N^0, v, \varphi\right) = \left\{\mathbf{u} \in \mathbb{R}^{n+1} \Big| \sum_{j\in N^0} u_j = v\left(N^0\right) \text{ and } \sum_{j\in S} u_j \geq v(S) \text{ for all } S \subseteq N^0\right\}.$$

An auction outcome is said to be in the core (or a core outcome) if the payoff profile of the outcome is in the core. An auction mechanism is said to be *core-selecting* (or said to have the *core property*) if all outcomes are in the core. Note that a core outcome must be efficient; otherwise, it is blocked by N^0. Although the core property is desirable, due to the impossibility result (Proposition 1), the following pairwise stability will be used as the alternative stability concept in this paper.

Definition 1. An auction outcome is said to be ***pairwise stable*** when there exists no blocking coalition that consists of the auctioneer and one bidder. An auction mechanism is said to be pairwise stable if the outcome is pairwise stable for any $T \in \mathcal{T}$.

By definition, if an outcome is not pairwise stable, it is not in the core. In our model, an outcome is pairwise stable if and only if $p \equiv \sum_{j\in N^0} p_j \geq b_{wj} + p_j$ for all $j \neq w$, $j \in N^0$. In particular, an LPF outcome is pairwise stable if and only if $p \geq b_{wj}$ for all $j \neq w$, i.e., when winner w wins at price p, no other bidder's bid against w is larger than p.

As an example of multidimensional direct auction mechanisms, the Vickrey-Clarke-Groves (VCG) mechanism is given.

Definition 2. For a reported type profile T, the VCG auction mechanism is $\varphi = (x, \rho)$, where $x(T) \in \arg\max_i \sum_j t_{ij}$ and $\rho_j(T) = \sum_{k\neq j} t_{x(T_{-j}),k} - \sum_{k\neq j} t_{x(T),k}$.

Example 1. For $T_N = \begin{bmatrix} 7 & 0 & -2 \\ -3 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $w = 2$, $\mathbf{p} = (0, 8, 1)$, and $p = 9$. There is a loser's payment. The outcome is not in the core, and is not even pairwise stable for bidder 1 (and the auctioneer) since $b_{21} = 10 > 9$.

Let $u_j(T'_S; T) = u_j(T'_S; T, \varphi)$ denote the utility of j when $S \subseteq N^0$, $0 \in S$, participates and submits T'_S to $\varphi = (x, \rho)$, while the true type of N^0 is T, i.e., $u_j(T'_S; T, \varphi) = t_{x(T'_S),j} - \rho_j(T'_S)$.

Definition 3. A mechanism φ is *incentive compatible* (IC)[17] for bidder j if

$$u_j\left(\left(\mathbf{t}_j,\mathbf{t}_{-j}\right);T\right)\geq u_j\left(\left(\mathbf{t}'_j,\mathbf{t}_{-j}\right);T\right)\text{ for all }T\in\mathcal{T},\mathbf{t}'_j\in\mathcal{T}_j.$$

A mechanism φ is *individually rational* (IR) for bidder j if

$$u_j\left(T;T\right)\geq v\left(j;T,\varphi\right)\text{ for all }T\in\mathcal{T}.$$

A mechanism φ is *weakly individually rational* (weakly IR) for bidder j if

$$u_j\left(T;T\right)\geq\inf_{T'_{-j}}u_j\left(T'_{-j};T\right)\text{ for all }T'_{-j}\in\mathcal{T}_{-j},T\in\mathcal{T}.$$

IR requires that the payoff of participation is at least the payoff of nonparticipation. Note that due to externalities, the right-hand side of the inequality is not zero. In contrast, weak IR requires that the payoff of participation is at least the "worst possible"[18] payoff of nonparticipation, i.e., $t_{wj}-p_j\geq\min_{i\neq j}\{t_{ij}\}$ in our model. Thus, weak IR is satisfied for losing bidders when there is no loser's payment. For the winner, weak IR requires $p_w\leq\max_{i\neq w}\{b_{iw}\}$, i.e., the payment is less than its maximum bid against all other bidders. As its name suggests, IR implies weak IR, but not vice versa. In the literature, due to the impossibility result in Myerson and Satterthwaite (1983), φ is said to be IC (or IR) if it is IC (or IR, respectively) only for all bidders $j\in N$, not including the auctioneer. However, weak IR can be satisfied for all bidders and the auctioneer, e.g., MSP.

3.2 Impossibility results

Hereafter, loser's payment-freeness (LPF) is given as the requirement of the mechanism design. In practice, a mechanism should have certain good properties. The core property is usually desirable since a blocking means "justified" envy of both the bidder(s) and the auctioneer ("low revenue"), and the outcome will be unstable if the blocking coalition actually deviates. Unfortunately, there is no LPF core-selecting mechanism. The intuition behind this is that the winner has to pay excessively to prevent a blocking, violating weak IR.

Proposition 1 (Jeong 2014)**.** *The VCG is neither core-selecting, pairwise stable, nor loser's payment-free. There exists no loser's payment-free core-selecting mechanism.*

[17] Throughout the definition, *ex-post* is assumed. Bayesian (or *ex-interim*) versions can be defined similarly. Note that ex-post IC and strategyproof are the same due to the independent private value assumption.

[18] In fact, if j does not participate, the outcome is fixed, i.e., φ runs an auction with T_{-j} and allocates to $x\left(T_{-j}\right)$. Thus, "worst possible" means the worst outcome out of any allocation to $N^0\backslash j$, i.e., when j does not participate, $N^0\backslash j$ chooses the worst outcome for j, which is the concept of α-effectiveness. That is, weak IR is IR by α-effectiveness (see Appendix B).

A mechanism, however, should still have a certain degree of stability and "fairness," and a well-known alternative to the core is pairwise stability (PS). If an outcome is not PS, it might be too disputable or easily deviated by the auctioneer and one bidder. In addition, as the core property resolves the low revenue problem, PS at least ensures the revenue of $\max_{j \neq w} \{b_{wj}\}$. Also, b_{j0} can be used as a reserve price for bidder j, which may be useful for government auctions. That is, the government can impose different reserve prices for each bidder (company) to mitigate monopoly or increase social welfare.

However, if we impose LPF and PS, neither incentive compatibility (IC) nor individual rationality (IR) is possible, even with inefficient mechanisms, as shown below. Again, the intuition behind this is the winner has to overpay, i.e., the winner regrets not losing. Note that PS implies that the auctioneer should sell when there exists $b_{0j} > 0$ for some bidder j. Thus, a trivial no-sale mechanism in inefficient cases cannot be a counterexample.

Proposition 2. *There is no mechanism that is loser's payment-free, pairwise stable, and also any of the following.*

(i) *incentive compatible*[19]

(ii) *individually rational*

(iii) *efficient and weakly individually rational*

Pairwise instability already implies the non-core property. The core should also be efficient and IR. Therefore, (ii) implies that a loser's payment-free core-selecting mechanism is impossible, as shown in Proposition 1. Moreover, (iii) shows that efficiency cannot be achieved if at least weak IR is required. That is, to achieve efficiency with LPF and PS, the winner may need to pay more than its maximum bid against all competitors.

These impossibility results might appear disappointing. However, when externalities exist, IR might be too strong a requirement, especially when predicting the outcome of nonparticipation is difficult. Nevertheless, auction mechanisms should have a certain degree of incentive to participate since we cannot simply make the winner arbitrarily overpay to achieve other desirable properties. In particular, it might be desirable that the payoff of participation is at least as large as the worst possible payoff of nonparticipation since this payoff is what bidders are guaranteed to receive with nonparticipation, which is weak IR. Thus, weak IR, which is in fact a well-known alternative to IR in the core with externalities literature (see Appendix B), is used as the participation constraint in this paper.

Hereafter, I impose LPF, PS, and weak IR as requirements. Even though IC is impossible, a mechanism should have certain good incentive properties, and *regret-freeness* of a certain kind in the following section can be considered strategyproofness with certain restrictions.

[19] For inefficient mechanisms, the following assumption is needed: the lowest type bidder cannot win unless every bidder reports the lowest type.

4 Ex-post regret

As shown in the previous section, there is no incentive compatible mechanism that is also pairwise stable and loser's payment-free. Some widely used mechanisms are not incentive compatible, but they try to minimize incentives to misreport in some ways. Likewise, ex-post "regret-freeness" can be considered an incentive property.

Under incomplete information, each bidder is typically assumed to be willing to maximize the expected payoff. Thus, a usual solution concept is some version of Bayesian equilibrium. If finding an equilibrium is difficult, however, it is unlikely that bidders will play an equilibrium. Moreover, a high-stakes auction is normally a rare event; thus, an optimal payoff in expectation might be less meaningful, one reason why alternative ex-post approaches exist, e.g., minimax regret (Savage 1951).

Since bidders are normally employees of companies, they might want to avoid certain undesirable outcomes that could be easily recognized by their CEOs or boards of directors. Obviously, due to the impossibility result on incentive compatibility, it is impossible for a mechanism to guarantee no regrets of any kind. Thus, we need to define the subset of regrets that does not occur by truthful report. That is, no regret of a certain kind means strategyproofness with certain restrictions.

For instance, in the first-price auction, winner w might regret not bidding lower. Likewise, in multidimensional auctions, w would regret not bidding $\mathbf{b}'_w$ if w could still have won at a lower price. Depending on the characteristics of $\mathbf{b}'_w$, this regret can be defined with different degrees in Definition 5. Overpay regret means the winner could have won at a lower price by a misreport, and other kinds of overpay regret have some restriction on misreports.

Definition 4. For a given $\mathbf{b}_j$ and $q \in \left[\underline{b}, \overline{b}\right]$, a ***q-capped bid*** is defined as $\mathbf{b}_j^{\overline{q}} \equiv \left(b'_{ij}\right)$, where $b'_{ij} = b_{ij}$ if $b_{ij} \leq q$, otherwise $b_{ij} = q$.

Definition 5. For mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, winner w has ex-post

- ***(anybid) overpay regret*** if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$
- ***underbid overpay regret*** if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$ with $\mathbf{b}'_w \leq \mathbf{b}_w$
- ***capped-bid overpay regret*** if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$ with $\mathbf{b}'_w = \mathbf{b}_w^{\overline{q}}$ for some $q \in \left[\underline{b}, \overline{b}\right]$

such that $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$ and $\rho_w(\mathbf{b}'_w, \mathbf{b}_{-w}) < p_w$.[20]

In other words, w has no overpay regret if $u_w(\mathbf{b}_w, \mathbf{b}_{-w}) \geq u_w(\mathbf{b}'_w, \mathbf{b}_{-w})$ for all $\mathbf{b}'_w$ and B when $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$. Note that without the restriction, "when $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$," the statement means strategyproofness for the winner, which is impossible. However, if we impose the restriction, there is a mechanism that is free of overpay regret, e.g., MSP.

[20] X (instead of x) is used to handle a tie. Also, for simplicity, let $\rho_w(\mathbf{b}'_w, \mathbf{b}_{-w})$ denote the payment of w when w wins by the tie-breaking. Note that "$\leq$" in "$\mathbf{b}'_w \leq \mathbf{b}_w$" is component-wise.

By construction, overpay regret includes underbid overpay regret,[21] and underbid overpay regret includes capped-bid overpay regret. For the characterization in Theorem 1, the weakest property, capped-bid overpay regret-freeness, is sufficient. However, MSP has the strongest property–overpay regret-freeness, i.e., the winner cannot win at lower price by any misreport, as shown in Theorem 2. Note that this extension is not obvious since a bid is multidimensional.

I will define "group winner regret" and "winner's regret" in Section 4.2. Due to the novelty and nontriviality of winner's regret and group winner regret, I will first introduce them informally with dynamic auctions where "drop" and "remaining" are well-defined. The intuition behind winner's regret is that the winner says, "I regret not dropping out at that price. No matter which remaining bidder had won at what price, I would have been better off." That is, the intuition behind *avoiding* winner's regret is, "I don't know who will win at what price if I drop out now. But I'm sure that I will be better off."

The direct analogue for group winner regret might seem to be that a group of bidders says, "We regret not dropping out together at that price. No matter which remaining nonmember had won at what price, *all of us* would have been better off." However, this (all of us) is too strong, and the intuition behind group winner regret is, "We regret not dropping out together at that price. No matter which remaining nonmember had won at what price, each member would have been better off than if that member or some other member had won." In other words, each member says, "Losing to any remaining nonmember is better than my winning, so I want to drop out. But I can't due to another member that I don't want to lose to," which induces "unnecessary" internal competition. When the group size is 2, all the members are better off losing to any remaining nonmember since the two members are all the members.

Now we need to interpret what "drop" and "remaining" mean in direct mechanisms. In direct mechanisms, a "drop" strategy can be implemented by a capped bid as the following obvious lemma.

Lemma 1. *For a given B and mechanism φ that is weakly IR, let $\varphi\left(\mathbf{b}_j^{\overline{q}}, \mathbf{b}_{-j}\right) = (w, \mathbf{p})$ for some $q \in \left[\underline{b}, \overline{b}\right]$. Then, it is impossible that $w = j$ and $p_w > q$. If φ is also LPF and PS, and there exists $i \neq j$ such that $b_{ji} > q$, then it is impossible that $w = j$ and $p_w \geq q$.*

"Remaining" bidders can be interpreted as "undominated" bidders in direct mechanisms, which will be defined in the following section by the two mechanisms φ^I and φ^{II} for winner's regret and group winner regret, respectively.

[21] Underbid overpay regret is related to the threshold price in Milgrom and Segal (2014).

4.1 Bid graph

I will first introduce the "bid graph," which is essential for the definition of group winner regret and the MSP mechanism itself. In fact, this graph representation can be used for standard one-dimensional auctions (as will be shown in Theorem 3). To the best of my knowledge, using a graph for the auction mechanism itself is a new approach.

Another advantage of a bid matrix is that it can be interpreted as an adjacency matrix for a directed weighted graph, a *bid graph*; i.e., each node represents a bidder, and each directed weighted edge represents a bid. Then, an English auction can be interpreted as updating a bid graph.

Definition 6. For a bid matrix $\widetilde{B} = \left(\tilde{b}_{ij}\right)_{i,j \in S \subseteq N^0}$, a directed weighted graph $G = G\left(\widetilde{B}\right) = (V, E, f)$ induced by $\widetilde{B}$ as an adjacency matrix is called a ***bid graph*** (or *social bid network*), where $V = \left\{j \in S | \tilde{b}_{ij} \neq 0 \text{ for some } i\right\}$ is a vertex set, $E = \left\{(i,j) \in S \times S | \tilde{b}_{ij} \neq 0\right\}$ is a directed edge set,[22] and $f(e) = \tilde{b}_{ij}$ for $e = (i,j) \in E$ is a weight function.

Notation. In Algorithms 1-3, initially $\widetilde{B} = B$, then $\widetilde{B}$ will be updated; i.e., $B = (b_{ij})$ denotes the original bid matrix, and $\widetilde{B} = \left(\tilde{b}_{ij}\right)$ denotes an updated bid matrix. $i \in G$ means $i \in V(G)$. $V(B) = V(G(B))$. $H \subseteq G$ denotes that H is a subgraph of G. In the algorithms, ":=" means "set" or "update," e.g., $\widetilde{B} \coloneqq B$ and $k \coloneqq k+1$.

Among four mechanisms that will be introduced in this paper, φ^I, φ^{II}, and MSP are direct mechanisms, and only ME is an open ascending dynamic mechanism. However, each of the three direct mechanisms works as a dynamic auction "internally" (or "proxy" dynamic auction); thus, for an intuitive explanation, the term "drop" will also be used for the direct mechanisms. In particular, "***drop*** bidder j" means "set $\tilde{b}_{ij} = 0$ for all i." Likewise, "***unblock*** $\tilde{b}_{ij}$" means "set $\tilde{b}_{ij} = 0$," and "j is blocking i at q" means "$\tilde{b}_{ij} > q$."

The rest of this section defines the two mechanisms, φ^I and φ^{II}, which were informally introduced in Section 2. φ^I and φ^{II} define (q,k)-*undominated bid matrix* and (q,k)-*group undominated bid matrix*, which denote the status of each auction at price q and step k, and will be used to interpret "remaining" bidders in terms of direct mechanisms for winner's regret and group winner regret, respectively. Note that (q,k) values are not necessary for the outcome itself. The first mechanism, φ^I, is a direct mechanism version of the English auction with the bidding function β^I (Eq. 2.1).

[22] In graph theory, for an adjacency matrix B to have full information of E and f, typically $b_{ij} = 0$ means no edge between i and j. Thus, for notational simplicity and alignment with notational conventions in graph theory, $\underline{b} > 0$ is assumed without loss of generality by Lemma 8. Note also that "$(i,j) \in S \times S$" is used instead of "$(i,j) \in V \times V$," the standard definition in graph theory. This allows us to describe the moment that some bidder i has just "dropped," i.e., $i \notin V$ ("*phantom node*"), while some bidder j was bidding against i, i.e., $(i,j) \in E$, e.g., between steps 5 and 6 in Algorithm 1 (see $(26,2)$ in Figure 2.1).

Algorithm 1. *Mechanism* φ^I

1. *Each bidder* j *submits* $\mathbf{b}_j \in \mathcal{B}_j$.[23] $\widetilde{B} \coloneqq B$. $\tilde{b}_{jj} \coloneqq 1$ *for all* j.[24] *Let* $G\left(\widetilde{B}\right) = (V, E, f)$.
2. $q \coloneqq \min\left\{\tilde{b}_{ij} | (i,j) \in E, i \neq j\right\}$. $k \coloneqq 0$ *and* $B_{(q,k)} \coloneqq \widetilde{B}$.
3. *(unblock)* $\tilde{b}_{ij} \coloneqq 0$ *for all* $(i,j) \in E$ *with* $\tilde{b}_{ij} = q$. $k \coloneqq k+1$ *and* $B_{(q,k)} \coloneqq \widetilde{B}$.
4. $D \coloneqq \left\{j \in V | \tilde{b}_{ij} = 0 \text{ for all } i \in V, i \neq j\right\}$. *If* $D = \emptyset$, *go to step 2.*
5. *(sequential drop)* $\tilde{b}_{jj} \coloneqq 0$ *and* $\left(q_j^I, k_j^I\right) \coloneqq (q,k)$ *for all* $j \in D$. $k \coloneqq k+1$ *and* $B_{(q,k)} \coloneqq \widetilde{B}$. *If* $|V| = 0$, *go to step 7.*
6. *(sequential unblock)* $\tilde{b}_{ij} \coloneqq 0$ *for all* $(i,j) \in E$ *with* $i \in D$. $k \coloneqq k+1$ *and* $B_{(q,k)} \coloneqq \widetilde{B}$.
7. *If* $|V| > 1$, *go to step 4. Otherwise, winner* w *wins at price* $p = q$, *where* $w \in V$ *if* $|V| = 1$, *or* w *is chosen out of* D *by the tie-breaking rule if* $|V| = 0$. $\left(q_w^I, k_w^I\right) \coloneqq (q,k)$

Roughly, φ^I works as follows. Each bidder submits $\mathbf{b}_j$. In step 2, the mechanism increases the current price q to find $\tilde{b}_{ij} = q$. In step 3, it unblocks $\tilde{b}_{ij} = q$, which implies that j no longer needs to bid against i above price q. In steps 4 and 5, if removing the edge results in $\tilde{b}_{ij} = 0$ for all $i \neq j$, then it drops bidder j, otherwise it goes back to step 2 to increase the current price. If bidder j drops, then in step 6, the bidders who bid against j no longer need to bid against j, which is why step 6 is called a *sequential unblock*. This may in turn lead to some other bidder's drop in step 5, which is why step 5 is called a *sequential drop*. That is, a sequential unblock and a sequential drop can be repeated without the increase of q. This process is repeated until one bidder, the winner, is left. If all remaining bidders simultaneously drop,[25] then one of them is chosen to be the winner. Only the winner pays, and the price is the last $q = b_{ij}$, the last bid that is unblocked in step 3.

Definition 7. $B_{(q,k)}$ is called a (q,k)***-undominated bid matrix***.[26] A bidder j is said to be *undominated* at (q,k) if $j \in V\left(B_{(q,k)}\right)$, otherwise *dominated* at (q,k). That is, $V\left(B_{(q,k)}\right)$ is the *set of undominated bidders* at (q,k).

[23]If the auctioneer has no reserve price, i.e., "must sell," then the auctioneer does not need to submit the bid, i.e., $\mathcal{B}_j$ is either $\mathcal{B}_{j,N}$ or $\mathcal{B}_{j,N^0}$ depending on this assumption. For simplicity, updating $\widetilde{B}$ in any of the subsequent steps also implies updating G, i.e., $G\left(\widetilde{B}\right) = (V, E, f)$ is induced again.

[24]For illustrative purposes, $\tilde{b}_{jj}$ is used to differentiate "right before" and "right after (i.e., phantom node)" a drop, and it does not create a self-loop in a bid graph, e.g., $(26,1)$ and $(26,2)$ in Figure 2.1. That is, even if $\tilde{b}_{ij} = 0$ for all $i \neq j$, bidder j does not drop if $\tilde{b}_{jj} = 1$, which is also useful to make the winner undropped.

[25]This is possible even without a tie in bids. For instance, suppose bidders 1, 2, and 3 are remaining bidders, and $\tilde{b}_{21}$, $\tilde{b}_{12}$, and $\tilde{b}_{13}$ are only bids that have not been unblocked yet, and $\tilde{b}_{21} < \tilde{b}_{12}$, $\tilde{b}_{21} < \tilde{b}_{13}$, and $\tilde{b}_{12} \neq \tilde{b}_{13}$. That is, bidders 2 and 3 are staying only due to bidder 1. Then, bidder 1 drops first, and then bidders 2 and 3 drop simultaneously through the sequential unblock and the sequential drop.

[26]For illustrative purposes only (if needed), e.g., to show the status of an auction at any price, for the (q,k) that is not used in the algorithm, $B_{(q,k)}$, where $q \in \left[\underline{b}, \overline{b}\right]$ and $k \in \mathbb{Z}_+ \cup \{\infty\}$, can be defined as the lexicographically previous entry. For instance, if $B_{(2,6)}$ and $B_{(4,0)}$ are the two consecutive undominated bid matrices used in the algorithm, then $B_{(2,6)} = B_{(2,k)} = B_{(3,k')} = B_{(4,0)}$ for $k \in [6,\infty]$, $k' \in [0,\infty]$. Also, $B_{(\underline{b},0)} \equiv B$. A similar extension can be applied to $B^{(q,k)}$ in Definition 8.

As shown in Figure 2.1, the (q,k)-undominated bid matrix denotes the status of an auction by φ^I at price q and step k. In particular, $\left(q_j^I, k_j^I\right)$ denotes the price and the step when bidder j drops (at step 5) or wins (at step 7). When B needs to be specified, $\left(q_j^I, k_j^I\right)$ will be written as $\left(q_w^I(B), k_w^I(B)\right)$. In fact, if j drops at $\left(q_j^I, k_j^I\right)$, then j's staying any longer is dominated (in English auctions), assuming no one has played dominated strategies. That is, losing to any bidder $i \in V\left(B_{\left(q_j^I, k_j^I\right)}\right) \setminus \{j\}$ is weakly better off than if j wins at any price $p \geq q_j^I$ (strictly better off if $p > q_j^I$). φ^I drops bidders who are dominated as prices increase.

The reason why group winner regret happens can be intuitively explained by the connectedness of a bid graph. To do this, I first introduce a series of definitions and notations, which were roughly explained in Section 2.

For a directed graph, a *path* is a sequence of nodes (including phantom nodes) that is connected by a sequence of edges. The length of a path is defined by the number of edges in the path. A *direct path* is a path that has a unit length. The following "arrow" notation, e.g., $i \to j$, will be used instead of the usual "sequence" notation, e.g., $(i, ..., j)$.

Notation. $i \to j$ denotes a path from i to j.[27] Likewise, $i \nrightarrow j$ denotes there is no path, and $i \rightleftarrows j$ denotes both $i \to j$ and $j \to i$. If $i \to j$ is a direct path, then it is denoted by $i \twoheadrightarrow j$.

A *strong component (or strongly connected component) of* G is a maximal subgraph H of G such that for any $i, j \in H$ with $i \neq j$, there exists a path between i and j in both directions. In other words, $H \subseteq G$ is a strong component if $i \rightleftarrows j$ for all $i, j \in H$ with $i \neq j$, and there is no $k \notin H$ such that $k \rightleftarrows i$ for some $i \in H$. Let $\mathcal{G}$ denote the set of strong components of G, i.e., $\mathcal{G} \equiv \{H \subseteq G | H \text{ is a strong component of } G\}$.

A *weak component* (or *weakly connected component*) of G is a maximal subgraph H of G such that for any $i, j \in H$ with $i \neq j$, there exists a path in the undirected graph induced by G. In other words, $H \subseteq G$ is a weak component if $i \to j$ or $j \to i$ for all $i, j \in H$ with $i \neq j$, and there is no $k \notin H$ such that $k \to i$ or $i \to k$ for some $i \in H$.

G is said to be *connected* (or *weakly connected*) if G has only one weak component, i.e., G itself is a weak component. G is said to be *strongly connected* if G has only one strong component, i.e., G itself is a strong component. G is said to be *only weakly connected* if it is connected, but not strongly connected.

In fact, each *strong component* can be interpreted as a "*group*." For instance, in Figure 2.2, at $(20, 1)$, group G_2 no longer needs to bid against group G_1 while G_1 still bids against G_2, which can be interpreted as the "unnecessary" internal competition of G_2. This is the main idea behind group winner regret and MSP. I first introduce φ^{II}, which can be greatly simplified into MSP by the Chain Lemma (Lemma 2), and define a (q,k)-group undominated

[27] "$i \to j$" denotes either a path itself or its existence, i.e., "in G, $i \to j$" is equivalent to "G has $i \to j$."

bid matrix, $B^{(q,k)}$, which will be used to define group winner regret. In fact, φ^{II} is a direct mechanism version of the English auction with the bidding function β^{II} (Eq. 2.2).

Algorithm 2. *Mechanism φ^{II} (equivalent to the MSP in Algorithm 3)*

1. *Each bidder j submits $\mathbf{b}_j$. $\widetilde{B} := B$. $\tilde{b}_{jj} := 1$ for all j. Let $G\left(\widetilde{B}\right) = (V, E, f)$.*[28]
2. *$q := \min\left\{\tilde{b}_{ij} | (i,j) \in E, i \neq j\right\}$. $k := 0$ and $B^{(q,k)} := \widetilde{B}$.*
3. *$M := \left\{j \in V | \tilde{b}_{ij} = q\right\}$. If $M = \emptyset$, go to step 2. Otherwise, choose a bidder $m \in M$.*[29]
4. *$C := \left\{i \in V | \tilde{b}_{im} = q\right\}$. If $C = \emptyset$, go to step 3. Otherwise, choose a bidder $c \in C$.*
5. *(unblock) $\tilde{b}_{cm} := 0$. $k := k+1$ and $B^{(q,k)} := \widetilde{B}$.*
6. *$\mathcal{G} := \{H \subseteq G | H$ is a strong component of $G\}$. If $|\mathcal{G}| = 1$, go to step 4.*
7. *(sequential drop) $\mathcal{D} := \left\{H \in \mathcal{G} | \tilde{b}_{ij} = 0 \text{ for all } j \in H, i \in K \in \mathcal{G}, K \neq H\right\}$. $\tilde{b}_{ij} := 0$ and $\left(q_j^{II}, k_j^{II}\right) := (q,k)$ for all $i \in V$ and $j \in D$, where $D = \cup_{H \in \mathcal{D}} V(H)$. $k := k+1$ and $B_{(q,k)} := \widetilde{B}$.*
8. *(sequential unblock) $\tilde{b}_{ij} := 0$ for all $(i,j) \in E$ with $i \in D$. $k := k+1$ and $B_{(q,k)} := \widetilde{B}$.*
9. *If $|V| > 1$, go to step 6. Otherwise, winner $w = c$ wins at price $p = q = b_{cm}$. $\left(q_w^{II}, k_w^{II}\right) := (q,k)$*

As explained in Section 2, roughly, φ^{II} works as follows. Each bidder submits $\mathbf{b}_j$. In steps 2-4, the mechanism increases the current price q to find $\tilde{b}_{cm} = q$. In step 5, it unblocks $\tilde{b}_{cm} = q$, which implies that m no longer needs to bid against c above price q. In step 6, if only one strong component (or group) exists, then it goes back to step 4 to find the next unblocking (if there is no one, go further back to step 2 to increase the price). Otherwise, i.e., if multiple groups exist, then in step 7, it finds groups that do not bid against some other group, i.e., $\mathcal{D}$ is the set of these groups. Then, it drops all bidders in these groups, i.e., D is the set of these bidders. If bidder j drops, then in step 8, the bidders who bid against j no longer need to bid against j (sequential unblock). If there still exist multiple groups, this leads to some other group's drop in step 7 (sequential drop). That is, sequential unblock and drop can be repeated without the increase of q. This process is repeated until one bidder, the winner (the last c), is left. Only the winner pays, and the winning price is the last $q = b_{cm}$, the last bid that is unblocked in step 5, which is defined as the ***second-price***.[30] The owner of this bid (the last m) is defined as the ***threshold bidder***, who made the auction end.

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[28] See footnotes 23 and 24.

[29] M stands for "Me" and C stands for "Competitor." "$M = \emptyset$" is possible when coming back from step 4. Likewise, in step 4, "$C = \emptyset$" is possible when coming back from step 6. Choosing $m \in M$ or $c \in C$ is done by a tie-breaking rule if needed. Note that steps 3 and 4 are separated to break all ties in C first once m is chosen so that φ^{II} can reduce to the second-price when there are no externalities (Theorem 3). Step 2 is separated from step 3 only to reset k whenever q changes. Thus, steps 2 and 3 are combined in MSP.

[30] The "multidimensional first-price" (MFP) auction can be easily implemented. There can be multiple candidates for the "first price," e.g., $p = \max_{i \in D}\{b_{iw}\}$ at step 9, i.e., the maximum bid against the remaining bidders right before the winning. Similar first-price extensions can be done for other mechanisms in the paper.

Definition 8. $B^{(q,k)}$ is called a (q,k)***-group undominated bid matrix***. A bidder j is said to be *group undominated* at (q,k) if $j \in V\left(B^{(q,k)}\right)$, otherwise *group dominated* at (q,k). That is, $V\left(B^{(q,k)}\right)$ is the *set of group undominated bidders* at (q,k).

As shown in Figure 2.2, the (q,k)-group undominated bid matrix denotes the status of an auction by φ^{II} at price q and step k. That is, φ^{II} drops a group of bidders who are group dominated as prices increase. In particular, $\left(q_j^{II}, k_j^{II}\right)$ denotes the price and the step when bidder j drops (at step 7) or wins (at step 9). $\left(q_S^{II}, k_S^{II}\right)$ denotes the "earliest" price and step among $\left(q_j^{II}, k_j^{II}\right)$ for all $j \in S$, i.e., $\left(q_S^{II}, k_S^{II}\right) = \left(\min_{j\in S}\left\{q_j^{II}\right\}, \min_{j'\in \arg\min_{j\in S}\left\{q_j^{II}\right\}}\left\{k_{j'}^{II}\right\}\right)$. Note that if $S = V(H)$ for some $H \in \mathcal{G}$, then $\left(q_j^{II}, k_j^{II}\right) = \left(q_{j'}^{II}, k_{j'}^{II}\right)$ for all $j, j' \in S$.

As in the undominated bid matrix, one might expect that if S drops at $\left(q_S^{II}, k_S^{II}\right)$, then S's staying any longer is dominated (in English auctions), assuming no one has played dominated strategies; i.e., every member $j \in S$ is (weakly) better off losing to any nonmember $i \in R = V\left(B^{\left(q_S^{II}, k_S^{II}\right)}\right)\backslash S$ than if some member $j' \in S$ wins at any price $q' \geq q_S^{II}$ (strictly better off if $q' > q_S^{II}$). Unfortunately, however, this is true only when $|S| \leq 2$, but not true in general. This is because a strong component does not imply that each pair of nodes has a *direct* bidirectional path, but only a bidirectional path, which can be either direct or indirect. Thus, if $b_{j'j} < b_{ij}$, then j can be better off losing to j' than losing to i.

When $|S| = 2$, however, strong connectedness means a bidirectional path between two nodes. Thus, all bidders $j \in S$ prefer to lose to any nonmember $i \in R$, which is also trivially true when $|S| = 1$. When $|S| > 2$, to make all bidders $j \in S$ prefer to lose to any nonmember $i \in R$, one additional condition is needed, as shown below.

Definition 9. B is said to have *correlated externalities* if in $G\left(B^{(q,k)}\right)$ for some (q,k), $j \rightleftarrows j'$ and $j \nleftarrow i$ implies $b_{j'j} \geq b_{ij}$.

The correlated externalities means that at some price q, if bidder j' is at least an "indirect competitor" of j, i.e., $j \leftarrow j'$, and if j is also at least an indirect competitor of j', i.e., $j \rightarrow j'$, but i is not even an indirect competitor of j, i.e., $j \nleftarrow i$, then j suffers greater externality due to j' than due to i. Roughly speaking, this implies "if you are my competitor, your competitor is my competitor as well." This correlation is plausible when externalities mainly come from local competition. For instance, consider an auction for an MLB posting, where two leagues, the American and the National, exist. Each team is more likely to bid higher against teams in the same league than the other.

Note that the following one commonly used externality structure in the literature also satisfies the correlated externalities: each bidder has only one strong competitor, i.e., for each bidder j, there exists bidder k_j such that $b_{k_j,j} > b_{ij}$ for all $i \neq k_j$, and $b_{ij} = b_{i'j}$ for all $i, j \neq k_j$ with $i \neq j$ and $i' \neq j$.

4.2 Ex-post winner's regret and group winner regret

Now I will define ex-post winner's regret and group winner regret. To the best of my knowledge, group winner regret has not been studied in the auction literature. Its individual version, winner's regret, is also a novel approach that handles the case when one bidder imposes externalities on multiple bidders in a unified manner.[31] A (q, k)-undominated bid graph, $G\left(B_{(q,k)}\right)$, and a (q, k)-group undominated bid graph, $G\left(B^{(q,k)}\right)$, will be used to define winner's regret and group winner regret, respectively.

Definition 10. For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, winner w has ex-post ***winner's regret*** if there exists $q < p_w$ such that

$b_{iw} \leq q$ for all $i \in V\left(B'_{(q^I_w(B'), k^I_w(B'))}\right) \setminus \{w\} \neq \emptyset$, where $B' = \left(\mathbf{b}^{\overline{q}}_w, \mathbf{b}_{-w}\right)$.[32]

The above definition interprets the intuition behind winner's regret in terms of direct mechanisms: if w submits $\mathbf{b}^{\overline{q}}_w$ ("drop"), then w will be better off no matter which bidder $i \in R = V\left(B'_{(q^I_w(B'), k^I_w(B'))}\right) \setminus \{w\}$ (R: "remaining bidders") will win at any price $q' \geq q$ since $b_{iw} \leq q < p_w$.

If w has winner's regret in φ, then q (satisfying the condition) should exist. Thus, when q needs to be mentioned explicitly, it will be written as "w has winner's regret at q in φ" (the same notation applies to group winner regret). The infimum of such q's is $q^I_w(B)$ as in the following proposition, which can be used to determine whether w has winner's regret or not by comparing with p_w; and this proposition will also be used to prove the relationship between winner's regret and group winner regret in Corollary 4.

Proposition 3. *For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, winner w has winner's regret in φ if and only if $w \neq \varphi^I_1(B)$ and $p_w > q^I_w$.*

Corollary 1. *If w has winner's regret,* $\inf_q \{q | w \text{ has winner's regret at } q\} = q^I_w$.[33]

When the infimum needs to be mentioned explicitly, it will be written as "w has winner's regret at $\underline{q^I_w}$ in φ" (the same notation applies to group winner regret). As a corollary of Proposition 3, φ^I is winner's regret-free. In addition, φ^I has the following good properties that can be proven in a way similar to that for Theorems 1 and 2.

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[31] Varma (2002) assumes that there is no such case, and Varma (1999) and Hu et al. (2013) study simple models with three bidders. This paper allows a general externality structure and any finite number of bidders, and defines winner's regret that can be avoided by truthful report.

[32] "$V(\cdots) \setminus \{w\} \neq \emptyset$" excludes the case that w still wins with B' at some $q' \leq q$, which is overpay regret. The same applies to group winner regret. Note also that p_w (instead of p) is used to include non-LPF mechanisms for the independence of the axioms in the characterization (Theorem 1).

[33] The infimum is necessary, i.e., the minimum may not exist. For instance, in Figure 4.1, if bidder 3 wins at some p such that $5 = q^I_3 < p \leq 9$, bidder 3 has winner's regret at $\underline{5}$. However, if it submits $\mathbf{b}^{\overline{q}}_3$ for $q = 5$, then $R = \{1, 2, 4\}$, not $R = \{4\}$ for $5 < q < p$. The same applies to group winner regret.

Proposition 4. *φ^I is loser's payment-free, pairwise stable, weakly IR, winner's regret-free, and overpay regret-free.*

However, φ^I is not a unique mechanism that satisfies all the axioms in the proposition. MSP, for instance, also satisfies all of them. Moreover, φ^I is not group winner regret-free.

One might claim that the definition of winner's regret in φ should not use an *external* reference mechanism, φ^I, i.e., the set of the "remaining" bidders, R, should not be defined by φ^I. This may seem a reasonable concern. However, note that we first need some restriction on R since it is impossible to have no regret against all other bidders. Thus, R should be some subset of $N^0 \backslash \{w\}$. The choice of $R = V\left(B'_{(q^I_w(B'),k^I_w(B'))}\right) \backslash \{w\}$ is reasonable for direct mechanisms for the following three reasons, which can also be applied to group winner regret similarly. First, there are mechanisms free of the regret, e.g., φ^I and φ^{II} (or MSP). Second, it has a well-founded interpretation in English auctions, i.e., R is the set of undominated bidders since φ^I drops dominated bidders successively as prices increase. Thus, while some $i \in R$ will definitely win at some $q' \geq q$ when w drops at q, any $i \in D = R^C \backslash \{w\}$ cannot win since it has been dropped already. That is, there is no reason that w worries if some $i \in D$ might win when w drops at q even when $b_{iw} > q$.

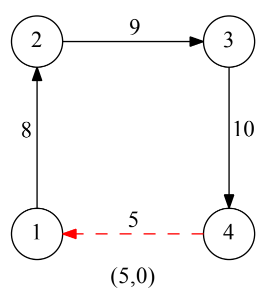

Figure 4.1: "remaining" bidders in direct mechanisms

Likewise, in direct mechanisms, if some $i \in D$ wins at some $q' \geq q$ when w submits $\mathbf{b}_w^{\overline{q}}$, the following problem occurs, which is the third justification for R. The problem is that now i has "winner's regret," i.e., with respect to the outcome for $B' = \left(\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w}\right)$, i regrets not using a q''-capped bid, where $q^I_w < q'' < q'$. In Figure 4.1, for instance, bidder 4 wins at 5 in φ^I, and $q^I_j = 5$ for all j. Suppose bidder 3 wins at 9 in φ, which is winner's regret at q, where $q^I_3 < q < 9$. If bidder 3 submits $\mathbf{b}_3^{\overline{7}}$, i.e., $B' = \left(\mathbf{b}_3^{\overline{7}}, \mathbf{b}_{-3}\right)$, then $R = \{4\}$ and $D = \{1,2\}$. Now, if bidder $i \in D$ wins at $q' \geq 7$ for B', the following problem occurs. First, bidder 1 cannot win at $q' \geq 7$ due to weak IR. However, if bidder 2 wins at $q' \geq 7$ (which is weakly IR), then bidder 2 regrets not losing to bidder 3 or 4 by using $\mathbf{b}_2^{\overline{q''}}$, where $q^I_2 < q'' < q'$, e.g., $q'' = 6$, since for $B'' = \left(\mathbf{b}_3^{\overline{7}}, \mathbf{b}_2^{\overline{6}}, \mathbf{b}_{\{1,4\}}\right)$, bidder 1 cannot win at $q' \geq 6$ due to weak IR. That is, roughly, winner's regret can be recursively defined without φ^I as follows:

w has winner's regret *if there exists $q < p_w$ such that $b_{iw} \leq q$ for all i that can win at q, and there does not exist $q' < q$ such that $b_{i'i} \leq q'$ for all i' that can win at q', and there does not exist $q'' < q'$ such that $b_{i''i'} \leq q'$ for all i'' that can win at q'', ...*

Note that the part "*all i that can win at q, ...*" restricts R, which makes $R = V\left(B'_{\left(q_w^I(B'),k_w^I(B')\right)}\right) \setminus \{w\}$ in Definition 10; e.g., in the above simple example in Figure 4.1, the recursion of the definition ends at the second line since there is no such i'. Group winner regret can be defined in a similar way using a group undominated bid graph.

Definition 11. For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, a group of bidders $S \subset N^0$ with $w \in S$ has ex-post ***group winner regret*** if there exists $q < p_w$ such that

(i) $b_{ij} \leq q$ for all $j \in S$ and $i \in V\left(B'^{\left(q_S^{II}(B'),k_S^{II}(B')\right)}\right) \setminus S \neq \emptyset$, where $B' = \left(\mathbf{b}_S^{\overline{q}}, \mathbf{b}_{-S}\right)$, and

(ii) for all $j, j' \in S$ with $j \neq j'$, there exists a sequence of distinct bidders $(j_n)_{n=1}^k$ such that $j_1 = j$, $j_k = j'$, $b_{j_{n+1},j_n} \geq q$ for all $1 \leq n \leq k-1$, and $b_{1,k} \geq q$.[34]

The above definition interprets the intuition behind group winner regret in terms of direct mechanisms: (i) if S submits $\mathbf{b}_S^{\overline{q}}$ ("drop"), then every member $j \in S$ will be better off, no matter which nonmember $i \in R = V\left(B'^{\left(q_S^{II}(B'),k_S^{II}(B')\right)}\right) \setminus S$ ("remaining bidders") wins at any price $q' > q$, than if j wins. However, (ii) each member j_1 is better off winning (at q) than losing to some other member j_2, and j_2 is better of winning than losing to j_3, ..., and j_k is better of winning than losing to j_1. That is, (ii) is the reason for the "unnecessary" internal competition. Also, (i) and (ii) imply that $b_{k_j,j} \geq b_{ij}$, which in turn means that any member j is better off losing to any nonmember i than to some other member k_j. The following proposition is the counterpart of Proposition 3.

Proposition 5. *For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, a group of bidders S with $w \in S$ has group winner regret in φ if and only if $w \neq \varphi_1^{II}(B)$, $p_w > q_S^{II}$, and $S = \left\{j | q_j^{II} = q_w^{II} \text{ and } k_j^{II} = k_w^{II}\right\}$.*

Corollary 2. *If S has group winner regret,* $\inf_q \{q | S$ *has group winner regret at* $q\} = q_S^{II}$.

Corollary 3. *φ^{II} is group winner regret-free.*

Example 2 (Motivating example continued)**.** For B in Section 2, $S = \{1, 2\}$ has group winner regret at $\underline{20}$ in φ^I since bidder 1 wins at $26 > q_S^{II} = 20$. S has group winner regret at $\underline{20}$ in the second-price (SP) auction as well. In SP, a unique symmetric Bayesian Nash equilibrium (Jehiel et al. 1999, Proposition 4) b_j^* is the average of b_{ij} for all $i \neq j$, i.e., $b_j^* = \overline{b_{ij}}$ for $i \neq j$. Thus, $\mathbf{b}^* = (16 + 12/2, 20 + 6/2, 21) = (22, 23, 21)$, and bidder 2 wins at $22 > q_S^{II}$.

[34] This ensures that H, where $S = V(H)$, has a bidirectional path between each pair of nodes. (i) ensures that H is a maximal subgraph that has this property. That is, (i) and (ii) together make H strongly connected. See also footnote 32.

Note that the winners in English and SP auctions are different, but both outcomes are inefficient. In any case, if either bidder 1 or 2 bids lower to try to avoid group winner regret, the result might be pairwise unstable depending on the bids of others. For instance, suppose $\mathbf{t}_2 = (0, 22, 0)^T$. If bidder 1 drops before bidder 3, then bidder 2 wins at 21, which is not pairwise stable by bidder 1 since $b_{21} = 28 > 21 = p$. That is, bidder 1 regrets not bidding high enough. Thus, the group winner regret problem cannot be solved individually.

As its name suggest, group winner regret includes winner's regret, i.e., if there exists winner's regret, then there exists group winner regret, but not vice versa. I will show this as a corollary of a more general result. One main difference between φ^I and φ^{II} is that φ^I drops bidders *individually* while φ^{II} drops bidders *groupwise*. Thus, intuitively, we can expect that φ^{II} drops each bidder no later than φ^I, which is in fact true.

Proposition 6. *For any B, $\mathbf{q}^I(B) \geq \mathbf{q}^{II}(B)$. That is, each bidder never drops at a higher price in φ^{II} than in φ^I.*

Corollary 4. *For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, if w has winner's regret at q in φ, then there exists $S \subset N^0$ with $w \in S$ such that S has group winner regret at q' in φ with $q' \leq q < p_w$.*

Proof. By Propositions 3, 5, and 6. □

5 Multidimensional second-price and English auctions

This section will present the multidimensional second-price (MSP) auction, which is a simplified version of φ^{II} in Algorithm 2. Before we proceed, I first introduce some definitions. A *component path* is a sequence of strong components such that there exists a path of edges from each component to the next component.

Notation. For strong components $G_1, G_2 \in \mathcal{G}$, $G_1 \to G_2$ denotes a component path, i.e., there exists a path $i \to j$ for some $i \in G_1$ and $j \in G_2$. $G_1 \twoheadrightarrow G_2$ denotes a direct component path. For strong components, a subscript denotes any general sequence of indices, but a superscript denotes the strong component that contains a certain bidder, e.g., G^j is the strong component that includes bidder j, i.e., $j \in V(G^j)$, which is simply denoted by $j \in G^j$.

A *component chain*[35] of G is a maximal component path with no repeated component, i.e., a maximal simple component path. In other words, a component path, $G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_k$

[35]In graph theory, there is no agreement on the term "chain." Some authors use it to mean *path*, and some prefer to mean *simple path*, where *simple* means *no repeated node (or component)*. Also, some use *path* in a *directed* graph and *chain* in an *undirected* graph. I use *chain* to mean *maximal simple path* in a directed graph for simplicity.

for some $k \geq 1$, is a component chain if all $G_i \in \mathcal{G}$, $1 \leq i \leq k$, are distinct, and there is no $H \in \mathcal{G}$ such that $H \twoheadrightarrow G_1$ or $G_k \twoheadrightarrow H$. A component chain is said to be *clean* if (1) there is no $H \in \mathcal{G}$ with $H \neq G_i$ for all i, $1 \leq i \leq k$, such that $H \twoheadrightarrow G_i$ or $G_i \twoheadrightarrow H$ for some i, $1 < i < k$, but (2) there may exist another component chain $G'_1 \twoheadrightarrow G'_2 \twoheadrightarrow \cdots \twoheadrightarrow G'_{k'}$ such that $G_1 = G'_1$ and $G_k = G'_{k'}$. G is said to be a *connected component chain* if G consists of only one or more clean component chains, and G is connected.[36]

Algorithm 3. *The Multidimensional Second-Price (MSP) Auction*

1. *Each bidder j submits $\mathbf{b}_j$. $\widetilde{B} := B$. $\tilde{b}_{jj} := 1$ for all j. Let $G\left(\widetilde{B}\right) = (V, E, f)$.*[37]
2. $q := \min\left\{\tilde{b}_{ij} | (i,j) \in E\right\}$ *and* $M := \left\{j \in V | \tilde{b}_{ij} = q\right\}$*, then choose a bidder $m \in M$.*
3. $C := \left\{i \in V | \tilde{b}_{im} = q\right\}$*. If $C = \emptyset$, go to step 2. Otherwise, choose a bidder $c \in C$.*
4. ***(unblock step)*** $\tilde{b}_{cm} := 0$.
5. $\mathcal{G} := \{H \subseteq G | H$ *is a strong component of* $G\}$*. If $|\mathcal{G}| = 1$, go to step 3.*
6. ***(drop step)*** *(drop all bidders in all strong components except for G^c)*
   $\tilde{b}_{ij} := 0$ *and* $\tilde{b}_{ji} := 0$ *for all $i \in V$ and $j \in D$, where $D = \cup_{H \in \mathcal{G} \setminus G^c} V(H)$.*
7. *If $|V| > 1$, go to step 2. Otherwise, winner $w = c$ wins at price $p = q = b_{cm}$.*

Roughly, MSP works as follows. Each bidder submits $\mathbf{b}_j$. As MSP increases current price q, it unblocks $\tilde{b}_{cm} = q$, which implies that m no longer needs to bid against c above q. If this unblocking results in multiple strong components (or groups), then MSP drops (all the bidders in) all groups except for G^c, the group including c. That is, the *drop step* combines the *sequential drop* and *sequential unblock* steps of Algorithm 2. This is repeated until one bidder, the winner (the last c), is left. Only the winner pays, and the winning price is the last $q = b_{cm}$, the last bid unblocked in step 4, which is defined as the *second-price*. The owner of this bid (the last m) is defined as the *threshold bidder*, who made the auction end. Several examples of MSP with comparisons to other mechanisms are shown in Example 3.

In contrast to φ^{II}, MSP drops (all the bidders in) all groups except for G^c in the drop step. As shown in the following Chain Lemma, when unblocking b_{cm} induces multiple groups, there exists a clean component chain $G^m = G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_k = G^c$,[38] which will be called an *externality chain*. Moreover, even if multiple chains exist, every chain starts at G^m (m: who just gave up bidding) and ends at G^c (c: whom m just gave up bidding

[36] The last connectedness condition ensures that the multiple clean component chains (if any) have the same start and end components. See Figure 5.2 for an example of multiple chains. At $(2,1)$, $5 \twoheadrightarrow 3 \twoheadrightarrow 2 \twoheadrightarrow 1$ and $5 \twoheadrightarrow 4 \twoheadrightarrow 1$ are clean component chains, and G is a connected component chain. At $(2,3)$, a component chain $3 \twoheadrightarrow 2 \twoheadrightarrow 1$ is not clean due to $4 \twoheadrightarrow 1$.

[37] See footnotes 23, 24, and 29. Lemmas 2, 9, and 4 show that MSP and ME (Algorithm 4) are well-defined. Note that in Algorithms 1 and 2, "$\tilde{b}_{jj} := 1$" is mainly used to create a (group) undominated bid graph. Thus, "$\tilde{b}_{jj} := 1$" is unnecessary here, but without this, "$|V| > 1$" should be written as "$|V| > 0$" in step 7, i.e., the winner also drops. Thus, for consistency, "$\tilde{b}_{jj} := 1$" is also used here.

[38] This is for illustrative purposes only. In fact, the chain with a length of 2, $G^m \twoheadrightarrow G^c$, is possible.

against). In φ^{II}, G^m drops first to avoid group winner regret, and G_2 (and G_2' if another chain $G^m \twoheadrightarrow G_2' \twoheadrightarrow \cdots \twoheadrightarrow G^c$ exists) needs to unblock and drop (sequential unblock and sequential drop) to avoid group winner regret, and so on, until only G^c is left. That is, φ^{II} drops groups in the sequence of this externality chain. However, by the Chain Lemma, MSP can simply drop all groups except for G^c since the chain ends at G^c anyway. This not only increases computational efficiency, but also simplifies the proofs.

Lemma 2 (Chain)**.** *In the MSP, if unblocking $\tilde{b}_{cm}$ leads to $|\mathcal{G}| > 1$ (i.e., in the beginning of the drop step), then there exists at least one component chain. Each component chain is clean, and it starts at G^m and ends at G^c. Also, G is a connected component chain.*

Corollary 5. *The mechanisms φ^{II} and MSP are equivalent, i.e., both produce the same winner and the price up to ties.*

I present below three main theorems: characterization, incentive, and generalization.

Theorem 1 (Characterization)**.** *The MSP is a unique (up to ties) direct mechanism that is*

(i) *loser's payment-free (LPF),*
(ii) *pairwise stable (PS),*
(iii) *weakly individually rational (weakly IR),*
(iv) *group winner regret-free, and*
(v) *capped-bid overpay regret-free.*

It can be shown that the axioms (i)-(v) in the above characterization are independent, i.e., each axiom is needed for the characterization. Also, there exists the following alternative characterization. (See Appendix C for additional characterization results with fewer axioms.)

Corollary 6 (Alternative characterization)**.** *Of all mechanisms that satisfy (i)-(iv), MSP is a unique (up to ties) minimum-revenue direct mechanism.*

Although Theorem 1 is involved to prove the characterization, it is relatively easy to show that MSP satisfies each axiom; thus, here I provide some intuition: LPF is obvious by the algorithm; PS is not as obvious as in φ^I because in MSP, bidder j can be about to drop at q even when there exists $b_{ij} > q$ for some $i \in G$. However, by Chain Lemma, we know that such an i is still being blocked by j when a chain appears, i.e., either $G^i = G^j$ or $G^i \twoheadrightarrow G^j$. Thus, all such i must drop earlier than or together with j, which is formalized in the Generalized Pairwise Stability (GPS) Lemma (Lemma 6). Then, PS is a corollary of the GPS Lemma; weak IR is obvious for losers by LPF. For the winner, the price is at most the winner's maximum bid against some other bidder because there exists some bid that has not been unblocked before the winning; group winner regret-freeness is shown by Corollary 3;

and capped-bid overpay regret-freeness will be shown in a stronger form in the next theorem, which shows that MSP has better incentive properties than those in the characterization.

Theorem 2 (Incentive properties)**.** *The MSP has the following incentive properties.*

(i) *("no (anybid) overpay regret" or "one price for one bidder") The winner cannot win at a different (lower or higher) price by a misreport.*

(ii) *("no overturn regret") A loser cannot be better off winning by a misreport.*

(iii) *The gain by a misreport is bounded by the maximum difference of externalities, i.e.,* $\max_{\mathbf{b}'_j}\left\{u_j\left(\mathbf{b}'_j,\mathbf{b}_{-j}\right)\right\}-u_j\left(B\right)\leq\max_{i\neq j,k\neq j}\left\{b_{ij}-b_{kj}\right\}$ *for all* $\mathbf{b}'_j\in\mathcal{B}_j$ *and* $B\in\mathcal{B}$*.*

Note that no overpay regret does not imply incentive compatibility. The winner can be better off losing to some bidder by a misreport in some cases. Likewise, in some cases, a loser can be better off losing to another bidder than to the current winner by a misreport. Now, I explain some intuition behind the proofs. Considering the fact that a bid is multidimensional, no overpay regret might seem surprising. However, it can be easily proven by PS since $p=b_{wh}$, where h is the threshold bidder, i.e., the price is some bidder's bid against the winner. Thus, w cannot win at any lower price than p by PS.

No overturn regret means that $p_l\geq b_{wl}$, where p_l is a possible winning price for l by a misreport. Then, $p_l\geq b_{wl}$ can be shown by two steps: $p_l\geq q_l^{II}$ and $q_l^{II}\geq b_{wl}$. Note that the second inequality holds by the GPS Lemma. The first inequality can be shown by the Blocking Lemma (Lemma 7), which roughly means that when bidder l drops at q_l^{II}, there exists some bidder i such that $b_{li}\geq q_l^{II}$, i.e., i is blocking l until l drops. Thus, l cannot win at any price lower than q_l^{II}. The next theorem shows that MSP naturally generalizes SP.

Theorem 3 (Generalization of SP)**.** *The MSP satisfies the following.*

(i) *The MSP is strategyproof for a bidder without externalities imposed by others.*

(ii) *The MSP reduces to the second-price auction when there are no externalities.*

Note that (i) is a corollary of Theorem 2-(iii). Then, (ii) can be shown by (i) and Green-Laffont-Holmstrom Theorem. In fact, (ii) can be easily shown by the nature of the algorithm itself; i.e., when there are no externalities, MSP drops bidders in the same sequence as in the English auction whose direct mechanism version is SP.

MSP is a direct mechanism; however, it "internally" runs a dynamic auction after receiving bids. Thus, it can be easily implemented as a dynamic open ascending auction, the multidimensional English (ME) auction. If we interpret an English auction as the so-called button auction model, the one-dimensional English auction provides only one button to each bidder. On the other hand, ME provides each bidder separate buttons for each competitor. Initially, all buttons are pressed, i.e., "$\tilde{b}_{ij}=1$." If the current price q reaches b_{ij}, then bidder

j releases button for bidder i, i.e., "$\tilde{b}_{ij} = 0$." In the following algorithm, "submitting a unblocking bid C" implies "releasing buttons for all $i \in C$." From this information, we can construct the bid graph of a specific time.

Algorithm 4. *The Multidimensional English (ME) Auction*

1. *Initialize* $\tilde{B}$ *with* $\tilde{b}_{ij} := 1$ *for all* i *and* j*. Let* $G\left(\tilde{B}\right) = (V, E, f)$.[39] $q := 0$.
2. *Wait for an "unblocking bid"* C *for a given time.*[40] *If no bidders submit* C*, then increase* q *by a given increment, and go to step 2. Otherwise, let* m *be the bidder who submitted* C *first.*
3. *If* $C = \emptyset$*, go to step 2. Otherwise, choose a bidder* $c \in C$ *and* $C := C \setminus \{c\}$.
4. *(unblock step)* $\tilde{b}_{cm} := 0$.
5. $\mathcal{G} := \{H \subseteq G | H$ *is a strong component of* $G\}$*. If* $|\mathcal{G}| = 1$*, go to step 3.*
6. *(drop step)* $\tilde{b}_{ij} := 0$ *and* $\tilde{b}_{ji} := 0$ *for all* $i \in V$ *and* $j \in D$*, where* $D = \cup_{H \in \mathcal{G} \setminus G^c} V(H)$.
7. *If* $|V| > 1$*, go to step 2. Otherwise, winner* $w = c$ *wins at price* $p = q$.

The following corollary (of Theorem 3) shows that ME generalizes the English auction.

Corollary 7 (Generalization of English)**.** *The ME satisfies the following.*

(i) *The ME is strategyproof for a bidder without externalities imposed by others.*[41]

(ii) *The ME reduces to the English auction when there are no externalities.*

Some examples of MSP are provided with comparisons to other mechanisms. For completeness, some (q, k)-group undominated bid graphs (as in φ^{II}) are shown.

Example 3. For the second-price (SP) auction, a unique symmetric Bayesian Nash equilibrium $\mathbf{b}_j^* = \overline{b_{ij}}$ (average) for $i \neq j$ is used.

(1) Motivating example (Figure 2.2)

$$B = \begin{bmatrix} 0 & 26 & 21 \\ 28 & 0 & 21 \\ 16 & 20 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	3	$(0, 0, 20)$	Y	Y	Y
VCG	3	$(0, 0, 10)$	N	N	Y
SP	2	$(0, 22, 0)$	N	N	Y

[39] In ME, a directed graph $G = (V, E)$ is sufficient since each $\tilde{b}_{ij} \in \{0, 1\}$. See also footnotes 23 and 29.

[40] An *unblocking bid* denotes a set of bidders that a bidder no longer bids against. Thus, the *truthful unblocking bid* C_j of bidder j is $C_j = \{i \in V \setminus \{j\} | b_{ij} \leq q\}$. When multiple bidders submit unblocking bids, one bid is chosen by a tie-breaking rule. Bidders may have different strategies depending on the details of an implementation, e.g., what bidders can observe (e.g., existence itself of an unblocking bid of other bidder, identity (owner) of it, contents (competitors of the owner) of it), how to handle a tie (e.g., whether an unblocking bid that was not chosen is still valid after the chosen unblocking bid is processed). Note that since this paper does not go in depth into strategic issues related to the dynamic version of MSP, only the possibility of these various implementations is mentioned.

[41] In ME, bidder j is IC (or strategyproof) if the payoff when j uses the truthful unblocking bid (footnote 40) all the time is at least as large as the payoff when j uses any different unblocking bid.

At $(q,k) = (20,1)$, G_1 and G_2 are only weakly connected and $G_2 \twoheadrightarrow G_1$. Thus, all bidders in G_2 should drop to avoid group winner regret.

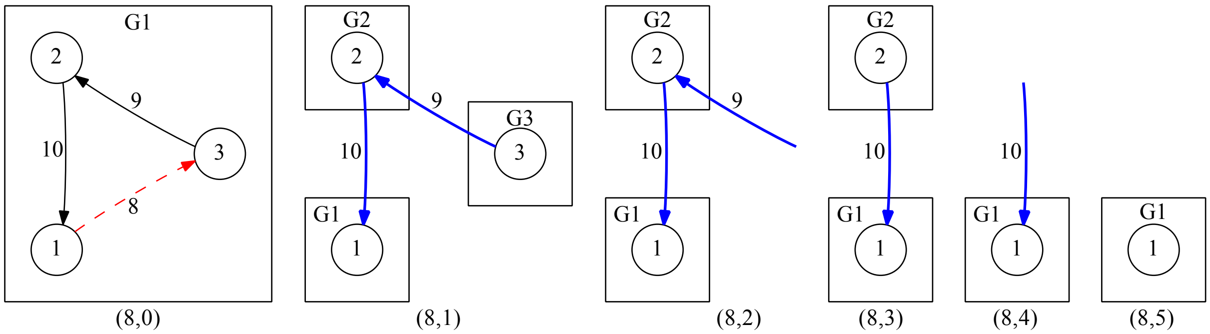

Figure 5.1: (q,k)-group undominated bid graph of the "cycle" example

(2) Cycle (Figure 5.1)

$$B = \begin{bmatrix} 0 & 5 & 8 \\ 10 & 0 & 5 \\ 5 & 9 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	1	$(8,0,0)$	N	Y	Y
VCG	1	$(8,3,0)$	Y	Y	N
SP	1	$(7,0,0)$	N	N	Y

This example shows that the pairwise comparisons of bids can induce a cycle. Bidder 1 beats bidder 2 by $b_{21} > b_{12}$, and bidder 2 beats bidder 3 by $b_{32} > b_{23}$, but bidder 3 beats bidder 1 by $b_{13} > b_{31}$. At $(q,k) = (8,1)$ in φ^{II}, unblocking $\tilde{b}_{cm} = \tilde{b}_{13}$ leads to a chain: $G^3 \twoheadrightarrow G^2 \twoheadrightarrow G^1$, and G is a connected component chain. As shown in the Chain Lemma, the end component is $G^c = G^1$. Thus, whereas φ^{II} drops G^3 first and then G^2, MSP drops them all at once. That is, in MSP, the next step of $(8,1)$ is $(8,5)$.

(3) Multiple chains (Figure 5.2)

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 3 & 0 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & 3 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	1	$(2,0,0,0,0)$	N	Y	Y
VCG	1	$(2,1,0,0,0)$	N	Y	N
SP	1	$(1.5,0,0,0,0)$	N	N	Y

At $(q,k) = (2,1)$, unblocking $\tilde{b}_{15}$ leads to two chains: $5 \twoheadrightarrow 3 \twoheadrightarrow 2 \twoheadrightarrow 1$ and $5 \twoheadrightarrow 4 \twoheadrightarrow 1$, and G is a connected component chain. As shown in the Chain Lemma, no matter which chain is chosen, the end component, G^1, is the same. Therefore, we can simply drop all the other components except for the end component to find the winner and the price. That is, in MSP, the next step of $(2,1)$ is $(2,7)$.

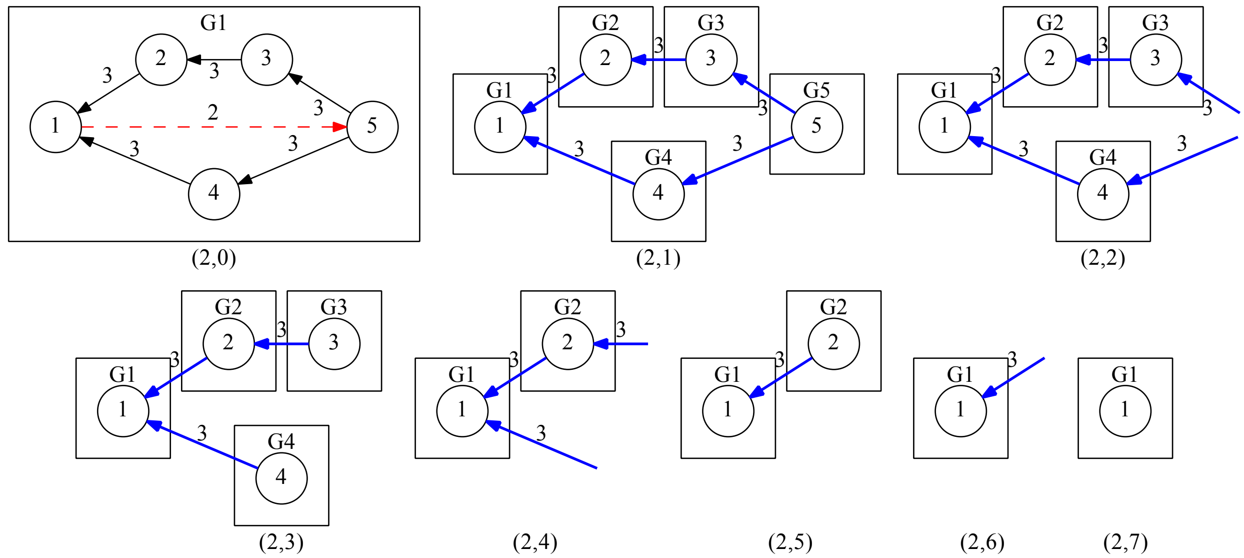

Figure 5.2: (q, k)-group undominated bid graph of the "multiple chains" example

(4) Problems of VCG in Example 1

$$B = \begin{bmatrix} 0 & 9 & 3 \\ 10 & 0 & 1 \\ 7 & 9 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	1	$(9, 0, 0)$	N	Y	Y
VCG	2	$(0, 8, 1)$	N	N	N
SP	2	$(0, 8.5, 0)$	N	N	Y

(5) Another problem of VCG (subsidy)

$$T = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 8 & 0 & 0 \\ 0 & 2 & 10 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right] \text{ or } B = \left[\begin{array}{c|ccc} 0 & 8 & 10 & 8 \\ \hline 0 & 0 & 10 & 8 \\ 0 & 6 & 0 & 1 \\ 0 & 8 & 10 & 0 \end{array}\right].$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	2	$(0, 6, 0)$	Y	Y	Y
VCG	2	$(0, -1, 0)$	N	N	Y
SP	2	$(0, 7, 0)$	Y	Y	Y

Now the auctioneer, bidder 0, is included to show a subsidy explicitly. Note that even though positive externalities exist, $b_{ij} > 0$ for all $i \neq j$ and $j \neq 0$. That is, even when all bids are positive, a subsidy occurs in VCG. Moreover, $t_{00} = 0$ cannot work as a reserve price. Thus, this is another example showing that VCG is prone to shill bidding.

6 Simulations

One might expect that group winner regret-freeness (GWRF) may increase efficiency since the existence of group winner regret implies that a group of bidders prefers another outcome. Likewise, pairwise stability (PS) may increase both revenue and efficiency. Due to multidimensionality, the analytic comparison of MSP and other mechanisms is difficult. In general,

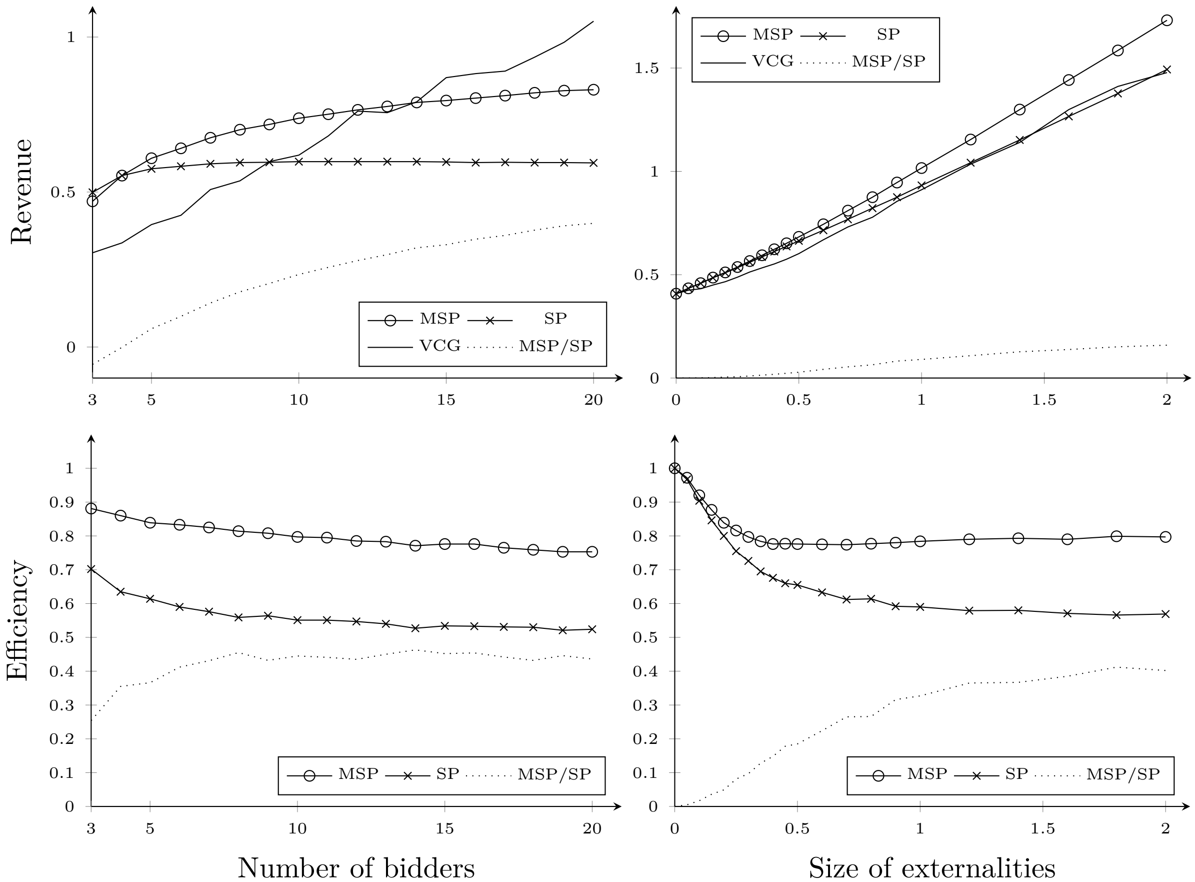

Figure 6.1: Main result on revenue and efficiency

for instance, neither MSP nor the second-price auction (SP) dominates the other in terms of either revenue or efficiency. Thus, I turn to simulations, which support the above intuition. Simulations suggest that MSP outperforms SP in terms of both revenue and efficiency.

For all simulations, the number of iterations is 5,000, and the following bidding strategies are used. For VCG, a truthful bid is used because it is incentive compatible. For SP, a unique symmetric Bayesian Nash equilibrium (Jehiel et al. 1999, Proposition 4), $b_j^* = \overline{b_{ij}}$ for $i \neq j$, is used. For MSP, a truthful bid is used. As in minimum-revenue core-selecting package auctions, due to several good incentive properties and multidimensionality, it seems quite difficult to manipulate MSP. Moreover, in some cases, a profitable manipulation $\mathbf{b}'_j \neq \mathbf{b}_j$ is only either underbid $\mathbf{b}'_j \leq \mathbf{b}_j$ or overbid $\mathbf{b}'_j \geq \mathbf{b}_j$; furthermore, in some other cases, a profitable manipulation $\mathbf{b}'_j \neq \mathbf{b}_j$ is neither overbid nor underbid, which makes a manipulation more difficult.

While I show the comparison of revenue and efficiency, I also examine the effects of the number of bidders and the size of externalities and therefore use a different model for each purpose. The first model is used to show the effect of the number of bidders, i.e.,

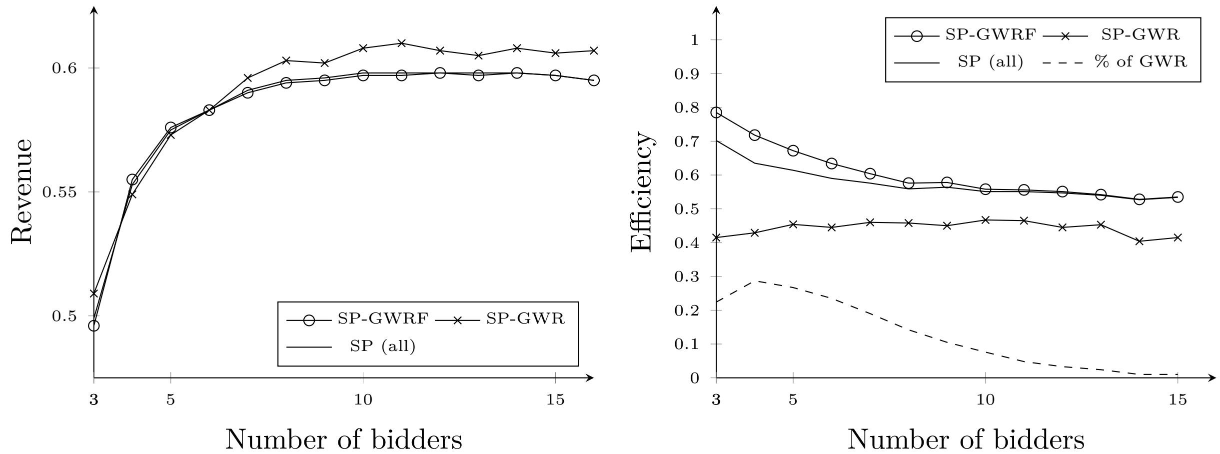

Figure 6.2: Effect of group winner regret-freeness (GWRF)

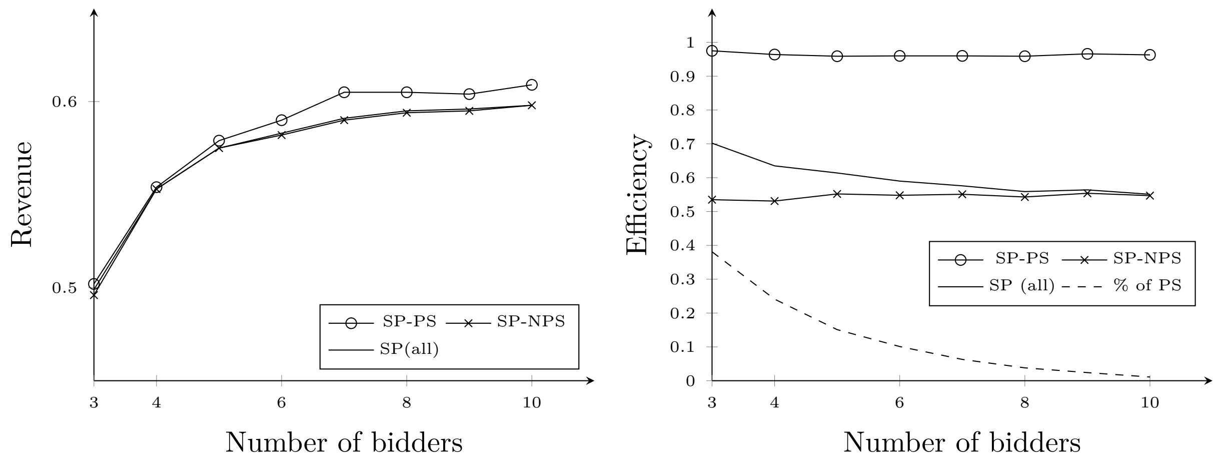

Figure 6.3: Effect of pairwise stability (PS)

x-value is n. Each bid is i.i.d. and drawn from a uniform distribution, i.e., $B_{ij} \sim U[0,1]$ for $i \neq j$. The second model is used to show the effect of the size of externalities when $n = 10$. Each valuation is i.i.d. and drawn from a uniform distribution, i.e., $T_{ii} \sim U[0, 0.5]$. Each externality is i.i.d. and drawn from another independent uniform distribution, i.e., $T_{ij} \sim U[-e, 0]$ for $i \neq j$, where e is the upper bound of the size of negative externalities, which is the value of the x-axis. Note that this model approaches the previous model (with a different bound) as the size of externalities increases because the externalities eventually dominate the valuation. Thus, all ratio values approach the same values of the first model with $n = 10$.

Figure 6.1 shows the main finding. The upper pane shows that MSP has higher revenue than SP for $n \geq 5$ on average.[42] Interestingly, the lower pane shows that MSP also has higher

[42] Because of the generality of the models, every comparison is regarding an average, not each instance.

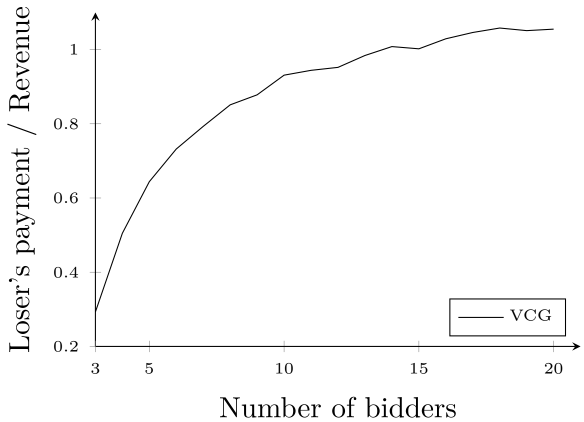

Figure 6.4: VCG's ratio of the loser's payment to revenue

efficiency than SP. That is, MSP outperforms SP in terms of both revenue and efficiency. "MSP/SP" denotes the gain, i.e., "MSP/SP (e.g., revenue) = (revenue of MSP / revenue of SP) - 1." The right pane shows that when there are no externalities, MSP and SP have the same outcome, as shown in Theorem 3. As externalities increase, MSP starts to outperform SP in terms of both revenue and efficiency, and the ratio values (i.e., efficiency and the gains of both revenue and efficiency) in the right pane approach the values in the left pane, as explained in the simulation model.

To examine the effect of GWRF or PS, I divide the outcomes. In particular, Figure 6.2 supports the intuition that GWRF tends to increase efficiency. It also supports that GWRF may decrease revenue (Proposition 6). "SP-GWRF" denotes the SP outcomes that are GWRF only, which can be found by Proposition 5. "SP (all)" denotes all SP outcomes, i.e., the same value as "SP" in Figure 6.1. "% of GWRF" denotes the percentage of the outcomes that are GWRF. On the other hand, Figure 6.3 supports the intuition that PS tends to increase both revenue and efficiency. "SP-(N)PS" denotes the SP outcomes that are (not) PS. Thus, for $n = 3$ in Figure 6.1, SP has higher revenue than MSP because the effect of GWRF might be larger than PS. However, any results for small n should be interpreted cautiously since the manipulation possibility of each mechanism might not be ignorable.

Interestingly, Figure 6.1 (upper pane) shows that, as n increases, VCG starts to have higher revenue than MSP around $n = 15$. However, Figure 6.4 shows that most of the revenue comes from losing bidders. In core-selecting mechanisms, the loser's payment ratio is higher than VCG. Thus, the loser's payment in VCG and core-selecting mechanisms might be a real concern. Note that the ratio can be larger than 1 due to a subsidy. Also, Figures 6.1 (lower pane) and 6.4 together suggest a tradeoff between efficiency and no loser's payment.

As mentioned earlier in this section, neither MSP nor SP dominates the other. Thus, I omit "on average" for the remainder of this section. Note also that the case of $n < 5$ will be explained in the next paragraph.

7 Conclusion

MSP is a unique direct mechanism that is pairwise stable, weakly individually rational, and free of a loser's payment, group winner regret, and capped-bid overpay regret. MSP sequentially satisfies a well-known alternative property in each category under the requirement of no loser's payment, i.e., pairwise stability instead of the core property and then weak IR instead of IR with the additional pairwise stability requirement. Regarding incentive properties, beyond the characterization result, MSP has further good incentive properties, and reduces to the second-price auction when there are no externalities.

As in the case of minimum-revenue core-selecting package auctions, several good incentive properties and multidimensionality seem to make manipulating MSP difficult. Kojima and Pathak (2009) show the asymptotic strategyproofness in many-to-one matching markets, and similarly, MSP might be asymptotically strategyproof in large markets. To prove or disprove this conjecture would be a good direction for future research. Likewise, because it is difficult to compare MSP to other mechanisms analytically, an experimental approach would be another good avenue for future research.[43]

Externalities are prevalent in auctions where commercial bidders participate. Even when externalities exist, the second-price auction is widely used. However, simulations suggest that MSP outperforms the second-price auction in terms of both revenue and efficiency. Also, in a general model, an optimal scalar bid in the second-price auction is not only difficult to calculate, but also prone to ex-post regret. In contrast, MSP is a direct mechanism that is free of certain kinds of regret. To the best of my knowledge, this paper is the first to introduce and resolve group winner regret. Moreover, MSP is the first mechanism that uses a network graph for an auction mechanism, which may inspire further research.

Thus far, even though multidimensional externality information exists, it has not been fully used in auctions. I believe one of the fundamental reasons for this is that no practical multidimensional mechanism for one item has been available.[44] This paper presents one candidate (and its open ascending version) that not only has good properties, but is also a natural multidimensional extension of the second-price (and English) auction.

[43]See Roth and Peranson (1999), Immorlica and Mahdian (2005), Bulow and Levin (2006), Che and Kojima (2010), Kojima and Manea (2010), Kojima et al. (2013), Azevedo and Budish (2013), Azevedo and Leshno (2013), Liu and Pycia (2013), Lee (2014), and Lee and Yariv (2014) for the large market literature. See also Budish and Cantillon (2012), Featherstone and Niederle (2013), and Fragiadakis and Troyan (2014) for some experimental studies.

[44]Although package auctions without externalities are already complicated both practically and theoretically (e.g., bidding language and finding an equilibrium bidding strategy), the extension of MSP into package or homogenous multi-item auctions is also an interesting future research direction.

A The MLB posting system

Since the Major League Baseball (MLB) posting system is an interesting real-world example of when MSP (or MFP in footnote 30 if "first-price" auctions are desired) or ME can be used as an alternative mechanism, I will explain it in more detail here.[45]

MLB is composed of two separate leagues, the American League (AL) and the National League (NL), and each league is further divided into three divisions, East, Central, and West. The 2014 MLB regular season consists of 162 games per team, and 20 of them are interleague games (prior to 1997, there were no interleague games). The postseason is as follows: Wild Card Game; AL and NL Division Series; AL and NL Championship Series; and World Series. One-dimensional auctions have been used for the MLB posting. Prior to 2013, the MLB Office of the Commissioner (MLB, hereafter) used a four-day-long first price (sealed-bid) auction for the posting fee with both KBO (Korea Baseball Organization) and NPB (Nippon Professional Baseball).[46] Starting in 2013, any MLB team can negotiate the contract, including the fixed ($20 million limit) posting fee suggested by an NPB team, with a posted Japanese player for 30 days, which is, in effect, a kind of an "open ascending auction."[47]

Externalities exist in the MLB posting auction. For instance, in 2012, Hyun-Jin Ryu from the KBO went to the LA Dodgers in the NL Western Division via the posting system. If he plays well for the Dodgers, the teams in the NL Western Division suffer the most negative externalities, and the teams in the NL Central or Eastern Division suffer the next most. The teams in the AL do not suffer that much unless there is a large chance that they face the Dodgers in the World Series, given the few and therefore insignificant interleague games. Therefore, each MLB team has a different maximum willingness to pay in order to beat each competitor.

This externality structure can make group winner regret a real concern. Suppose the LA Dodgers, the SF Giants, and the Boston Red Sox participate in a posting auction. Since both the Dodgers and the Giants are in the same NL Western Division and also rivals, they do not want to lose to each other; however, they might be willing to lose to the Red Sox in the AL above a certain price. If they fear losing to each other too much due to the so-called

[45] In fact, most points mentioned here are applied to other applications as well; however, I use the MLB posting example since it has some recent relevant changes and data that I can access. Any facts on MLB without references can be found at `www.mlb.com`.

[46] See `www.koreabaseball.com/FILE/ebook/pdf/2013regulation.pdf` (p. 151-157), link to "United States - Korean Player contract agreement" and `www.jpbpa.net/up_pdf/1284364663-401673.pdf`, link to "United States - Japanese player contract agreement" (both accessed January 19, 2015).

[47] See `m.mlb.com/news/article/66013956/mlb-npb-reach-agreement-on-posting-system`, link to "MLB, NPB reach agreement on posting system" (accessed January 19, 2015). MLB has not changed the agreement with KBO yet.

"Dodgers-Giants rivalry," then even if the Dodgers win, for instance, it might be possible that the Dodgers will pay too much and the Giants will suffer a large negative externality due to the Dodgers being in the same division and league.

Despite the presence of externalities, no loser's payment seems to be desired. No posting system with a loser's payment has ever been used. Of course, there might have been a secret side payment, either money or player in a future trade agreement, but no such payment has been revealed. Thus, the core property might not be necessary since it may require a loser's payment, and pairwise stability might be sufficient and important. It is hard to imagine that a losing team with a bid higher than the winning bid thinks the system is "fair." The auctioneer (MLB), the team of the posted player that will receive the posting fee, and the player who will receive the salary from the contract may also not want such a system even if it has a more efficient outcome, or a better outcome in some senses. Likewise, weak IR might be sufficient as well as desirable. It is difficult for each team to predict which team will win if it does not participate; however, it is undesirable that the winning team pays more than the maximum willingness to pay against all competitors.

Moreover, "perfect" efficiency might not be important since it does not consider MLB fans or others. In particular, once the reserve price is met, the posting right should be sold even if no-sale is efficient. Of course, higher efficiency is still desirable *ceteris paribus*. On the other hand, higher revenue is desirable and might be the most important objective in general. Although any high-stake contract in professional sports is prone to the "winner's curse," no group winner regret may mitigate the winner's curse. Therefore, the MLB posting system is a good real-world example that shows the potential use of MSP or ME.

In fact, there has been much criticism of the old system. The most important critique might be that it is too disadvantageous to a player. Note that the old system is an auction for exclusive negotiation rights, not including the contract with the player. Only the winner (a MLB team) can negotiate with the player after the posting auction. If the negotiation on the contract with the player fails, then the posting is canceled. The MLB team does not need to pay the posting fee (winning price), and the player must wait a year for the next posting. This might be one reason why the negotiation with the player after the posting auction tended to be significantly delayed.[48]

There have also been cases in which the first-highest bid for the posting fee has been significantly higher than the second-highest bid. For instance, the winning bid for Daisuke Matsuzaka in 2006 by the Red Sox was $51.1 million, which was roughly $11 million more

[48] In 2006, ESPN said, "maybe this was the most obvious game of chicken ever," see `sports.espn.go.com/mlb/columns/story?columnist=kurkjian_tim&id=2697354`, link to "Posting process needs to be altered" (accessed January 19, 2015).

than the second-highest bid.[49] If a winning bid for the posting fee is too high, then it may result in a lower salary for the player afterward, considering the fact that only the winner can negotiate with the player, and if the negotiation fails, the player must wait one more year. Note that the new system has a \$20 million limit for the posting fee. I suspect that this significant difference between the first-highest bid and the second-highest bid in the old system might partly come from the lack of a bidding language that can express the externalities. In contrast, the new system is a kind of "English auction." The standing offer can never decrease because any offer cannot be retracted normally. Thus, the new system is similar to "English auction with reentry" (Izmalkov 2003) since an outbid team can resubmit a new offer. In addition to openness, reentry may mitigate the problems of externalities.

It is too complicated to compare the new system with either MSP or ME due to multidimensionality in open auctions, i.e., finding an equilibrium is intractable in general. However, I still believe that in the new negotiation process, an MLB team may want to express a different willingness to pay depending on the current standing winner, i.e., they might want to retract its offer depending on the current standing winner, and this may complicate and delay the process even if each team knows its type, $\mathbf{b}_j$ in our model. Thus, ME (or MSP if sealed-bid is desired), which can directly utilize their types, might be a good alternative that simplifies and speeds up the posting process.

B The core with externalities

The core and individual rationality are defined by the coalition value function (or characteristic function). Due to externalities, we need to define the coalition value function carefully because the coalition value of a coalition without the auctioneer can be non-zero. To define any coalition value function, we need to choose what strategies the outside players, $S^C = N^0 \backslash S$, would use, which is the concept of *effectiveness* in the core with externalities literature. Of course, we can expect S^c to have an efficient outcome for the subtype T_{S^c} when side payments are allowed. That is, $v_\epsilon(S) = \begin{cases} \max_{i \in S} \sum_{j \in S} t_{ij} & 0 \in S \\ \sum_{j \in S} t_{w_{-S},j} & 0 \notin S \end{cases}$, where $w_{-S} \in \arg\max_{i \in S^C} \sum_{j \in S^C} t_{ij}$, which I will refer to as the coalition value function by ϵ-effectiveness (ϵ from "efficient").

However, ϵ-effectiveness might not be justified for auctions where side payments are prohibited because the core property is used as the concept of stability and "fairness" in auctions. In inefficient auctions with no side payments, a bidder would not decide to forgo participation, expecting an efficient allocation in her absence. Likewise, a bidder cannot reasonably claim that the auction outcome is unfair by comparing the realized payoff to

[49] See the same article in footnote 48.

the payoff she would receive if she had not participated and an efficient outcome had been realized. This is why there are multiple debatable concepts of effectiveness and corresponding core: α, β, and γ-core (Aumann and Peleg 1960; Shapley and Shubik 1969; Chander and Tulkens 1997).

In general, the coalition value functions by α and β-effectiveness (denoted by v_α and v_β, respectively) are defined as follows. Let Σ_j be the strategy set of player $j \in N^0$, and $\Sigma_S = \prod_{j \in S} \Sigma_j$ for $S \subseteq N^0$. Then, $v_\alpha(S) = v_\alpha(S; T) = \max_{\sigma_S \in \Sigma_S} \min_{\sigma_{-S} \in \Sigma_{-S}} \sum_{j \in S} u_j(\sigma_S, \sigma_{-S})$, and $v_\beta(S) = v_\beta(S; T) = \min_{\sigma_{-S} \in \Sigma_{-S}} \max_{\sigma_S \in \Sigma_S} \sum_{j \in S} u_j(\sigma_S, \sigma_{-S})$, where $u_j(\sigma_S, \sigma_{-S})$ is the utility of j when the coalition S play σ_S and the outsiders S^C play σ_{-S}. In our model, for S such that $0 \in S$, $\Sigma_{-S} = \emptyset$ (an auction needs the auctioneer) and $\Sigma_j = \mathcal{T}_j$ for $j \in S$. Therefore, in our model, α-core and β-core are the same.

Lemma 3. *α-core and β-core coincide, i.e. $v_\alpha(S) = v_\beta(S)$ for all $S \subseteq N^0$.*

Proof. If $0 \in S$, then $\Sigma_{-S} = \emptyset$. Thus, $v_\alpha(S) = v_\beta(S) = \max_{i \in S} \sum_{j \in S} t_{ij}$. Likewise, if $0 \notin S$, then $\Sigma_S = \emptyset$. Thus, $v_\alpha(S) = v_\beta(S) = \min_{i \in S^c} \sum_{j \in S} t_{ij}$. □

However, α-effectiveness is too "pessimistic," especially in auctions since it implies S^c makes the worst outcome for S. It is much more reasonable that players in S^c play for their own benefit, which is the concept of γ-effectiveness. In particular, γ-effectiveness assumes that players in S^C do not take particular coalitional actions against S, but play best responses individually. However, if finding the best individual strategy is difficult, the concept of γ-core becomes impractical.

One more reasonable alternative might be the following effectiveness such that for the coalition value of a coalition without the auctioneer, it is assumed that all other outside players run and participate in an auction.

Definition 12. For $S \subseteq N^0$, the *coalition value function* by *δ-effectiveness with respect to a mechanism* $\varphi = (x, \rho)$ is defined as

$$v_\delta(S) = v_\delta(S; T, \varphi) = \begin{cases} \max_{i \in S} \sum_{j \in S} t_{ij} & 0 \in S \\ \sum_{j \in S} t_{x(T_{-S}),j} & 0 \notin S \end{cases}.$$

As in $v_\delta(S; T, \varphi)$, the core should be defined with respect to φ, i.e., $Core(N^0, v, \varphi)$. Note that IR by "X"-effectiveness can be written as $u_j(T; T) \geq v_X(j)$, e.g., IR by δ-effectiveness can be written as $u_j(T; T) \geq v_\delta(j) = u_j(T_{-j}; T)$, and weak IR (IR by α-effectiveness) can be written as $u_j(T; T) \geq v_\alpha(j) = \min_{T'_{-j}} u_j(T'_{-j}; T) = \min_{T_{-j}} u_j(T_{-j}; T)$.

Another reason why "$v_\delta(S) = \sum_{j \in S} t_{x(T_{-S}),j}$ for $0 \notin S$" but "$v_\delta(S) = \max_{i \in S} \sum_{j \in S} t_{ij}$ for $0 \in S$" is reasonable is that the core only considers a one-step deviation. That is, when

S, $0 \notin S$, decides not to participate, S^C would "deviate" from nonparticipation, i.e., S^C would run an auction and participate. If the auction outcome is not efficient among S^C, then $S' \subseteq S^C$ might deviate once more, but this is the second-step deviation.

Note that ϵ-effectiveness is a special case of δ-effectiveness when the mechanism is efficient. Note that in this paper, the core and individual rationality are only used for the impossibility result, and the result holds with both δ- and e-effectiveness.

C Additional characterizations

MSP is a unique minimum-revenue direct mechanism that is LPF, PS, weakly IR, and GWRF (Corollary 6). Here I show two additional characterizations with fewer axioms. Note that the existence of the two mechanisms is proven by the existence of MSP that satisfies LPF, PS, weak IR, and GWRF.

Proposition 7. *Up to ties, the following holds.*

(i) *Of all mechanisms satisfying LPF and PS, a unique minimum-revenue direct mechanism is* $\varphi^{LP} = (x, \rho)$*, where* $x(B) \in \arg\min_i \max_{j \neq i} \{b_{ij}\}$*,* $\rho_w(B) = \max_{j \neq w} \{b_{wj}\}$*, and* $\rho_i(B) = 0$ *for all* $i \neq w$*.*

(ii) *Of all mechanisms satisfying LPF, PS, and weak IR, a unique minimum-revenue direct mechanism is* $\varphi^{LPW} = (x, \rho)$*, where* $x(B) \in \arg\min_i \max_{j \neq i} \{b_{ij}\}$ *such that* $\max_j \{b_{ij}\} \leq \max_j \{b_{ji}\}$*,* $\rho_w(B) = \max_{j \neq w} \{b_{wj}\}$*, and* $\rho_i(B) = 0$ *for all* $i \neq w$*.*

Proof. Both in (i) and (ii), LPF clearly holds. Minimum revenue and uniqueness hold by construction: each mechanism finds b_{ij} that satisfies the axioms in ascending order of b_{ij}. Thus, any different outcome with lower revenue is impossible.

(i) By PS, for i to win, $p_i \geq \max_{j \neq i} \{b_{ij}\} \equiv q_i$. Thus, the winner that makes the minimum revenue is $w \in \arg\min_i \{q_i\}$, and the price is $p_w = q_w$.

(ii) The constraint "$\max_j \{b_{ij}\} \leq \max_j \{b_{ji}\}$" means $q_i \leq \max_{j \neq i} \{b_{ji}\}$, which implies $q_w \leq \max_{j \neq w} \{b_{jw}\}$. Thus, φ^{LPW} is weakly IR for w. □

Note that the two mechanisms, φ^{LP} and φ^{LPW}, are different, and this is not a contradiction. That is, $p^{LP} \leq p^{LPW}$, where p^{LP} and p^{LPW} are the revenues of the two mechanisms, respectively. The inequality can hold strictly for some B. Likewise, $p^{LP} \leq p^{LPW} \leq p^{MSP}$, and the last inequality can hold strictly. Note also that each price is the greatest lower bound of the price of the mechanisms that satisfy each set of axioms, and this prevents "too low" revenue.

D Proofs

D.1 Proof of Proposition 2

Although the proposition statement is written as ex-post (IC or IR) for simplicity, it also holds for ex-interim (Bayesian) (IC or IR). The proofs for ex-interim versions are obvious from ex-post versions since a simple distribution can be easily found such that the same counterexamples hold. For simplicity, only bidders' types are shown in T.

(i) Since the assumption in footnote 19, i.e., the lowest type bidder cannot win unless every bidder reports the lowest type, is needed only for inefficient mechanisms, I will show efficient and inefficient cases separately.

efficient mechanisms Let $T = \begin{bmatrix} 7 & 0 & -2 \\ -3 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv T^1$ (Example 1). Since the payoff function is quasilinear, the only efficient and incentive compatible mechanism is VCG by the Green-Laffont-Holmstrom Theorem (Green and Laffont 1977; Holmstrom 1979; Walker 1978). In particular, since $\mathcal{T}$ is convex ($\mathcal{T}$ is closed and bounded in $\mathbb{R}^n$) and u_j is quasilinear, by Theorem 2 of Holmstrom (1979), the only efficient and incentive compatible mechanism is Groves' scheme.[50] Then, as shown in Example 1, VCG has a loser's payment, $\mathbf{p} = (0, 8, 1)$. Note that adjusting the payment for the lowest type (e.g., $\mathbf{p} = (-1, 7, 0)$ by lowering 1 for each bidder) cannot achieve LPF without a subsidy. (Even if we allow a subsidy, it is still not pairwise stable since $p_2 \geq \max_{j \neq 2} \{b_{2j}\} = 10$ is needed.)

inefficient mechanisms Let $T = \begin{bmatrix} 7 & -3 & -3 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \equiv T^2$. Without loss of generality, let $\underline{b} = 0$. Due to symmetry, we only need to check bidders 1 and 2. For bidder 1 to win, $p_1 \geq 8$, and then $u_1 \leq -1$. If bidder 1 submits a zero bid $(0, ..., 0)$ instead, either bidder 2 or 3 should win, and then the new utility $u_1' = 0 > u_1$. Thus, it is not IC for bidder 1. Now, for bidder 2 to win, $p_2 \geq 7$, and then $u_2 \leq -2$. If bidder 2 submits a zero bid instead, either bidder 1 or 3 should win. Note that bidder 1 can still win since neither efficiency nor IR is required. However, if bidder 1 wins, it is not IC for bidder 1 as before; therefore, bidder 3 should win, and $u_2' = 0 > u_2$. Thus, it is not IC for bidder 2. Note that the assumption

[50] Alternatively, Theorem 1 of Holmstrom (1979) can be directly used for the proof by showing that $\mathcal{T}$ is *smoothly connected*, which means that for any $\mathbf{t}_j, \mathbf{t}_j' \in \mathcal{T}_j$, there exists a one-dimensional parametrized family of valuation functions $\{v_j(i; y_j) \in \mathcal{T}_j | y_j \in [0,1]\}$, $i \in N$, $y_j \in [0,1]$ such that $v_j(i;0) = t_{ij}$, $v_j(i;1) = t_{ij}'$, and $\partial v_j(i; y_j) / \partial y_j$ exists for all $y_j \in [0,1]$. Note that $v_j(i; y_j) = t_{ij} + (t_{ij}' - t_{ij}) y_j$ satisfies the condition.

that the lowest type bidder cannot win is necessary. Otherwise, there exists an (inefficient) incentive compatible mechanism, e.g., a mechanism that makes bidder 1 always win with the minimum price that satisfies PS.

(ii) Let $T = T^2$. Due to symmetry, we only need to check bidders 1 and 2. For bidder 1 to win, $p_1 \geq \max_{j \neq 1} \{b_{1j}\} = 8$, and then $u_1 \leq -1 < \min_{i \neq 1} \{t_{i1}\} = 0$. Thus, it is not weakly IR for bidder 1. Also, for bidder 2 to win, $p_2 \geq \max_{j \neq 2} \{b_{2j}\} = 7$, and then $u_2 \leq -2$. If bidder 2 does not participate, either bidder 1 or 3 should win. In fact, if we use IR by ϵ-effectiveness, then bidder 3 should win; however, I also test bidder 1 to show the impossibility with other effectiveness, including IR by δ-effectiveness. Note that bidder 1 can still win since the outcome need not be efficient. However, if bidder 1 wins, it is not weakly IR for bidder 1 as explained before due to $b_{13} = 8$. Therefore, bidder 3 should win, and then the new utility $u_2' = t_{32} = 0 > u_2$. Thus, it is not IR for bidder 2, but weakly IR. Note that there exists a weakly IR mechanism, e.g., MSP (the mechanism of this paper).

(iii) Let $T = T^1$. By pairwise stability, $p_2 \geq \max_{j \neq 2} \{b_{2j}\} = 10$, then $u_2 \leq -1 < \min_{i \neq 2} \{t_{i2}\} = 0$. Thus, it is not weakly IR for bidder 2.

D.2 Proof of four main lemmas

Here I introduce the Connectedness, Generalized Pairwise Stability (GPS), and Blocking Lemmas and prove these as well as the Chain Lemma. One fundamental difference between English (or φ^I) and MSP auctions is that the bid graph of MSP is always connected as follows.

Lemma 4 (Connectedness)**.** *In the MSP, G is connected. In particular, G is strongly connected except for, possibly, step 5 (after the unblock step and before the drop step). That is, the unblock step results in either of the following two cases: (1) G remains strongly connected; (2) G is only weakly connected.*

Proof. G is strongly connected in the beginning of an auction. Also, each drop step leaves only one strong component (unless G is empty, which is also vacuously strongly connected), which means G is strongly connected. Thus, we only need to check the unblock step. Suppose G is disconnected after unblocking b_{cm}. Then there exists $k \geq 2$ strong components that are disconnected from each other. Now, adding back only one edge b_{cm} to G cannot make G strongly connected, which contradicts the strong connectedness of G before any unblocking. Therefore, G is at least weakly connected. The existence of both cases (1) and (2) can be shown by examples. □

To prove the Chain Lemma, I first prove the following simple lemma.

Lemma 5. *In the MSP, if unblocking b_{cm} leads to $|\mathcal{G}| > 1$, then $G^m \neq G^c$.*

Proof. Suppose $G^m = G^c$. Then there exists a strong component $H \in \mathcal{G}$ such that $H \neq G^m$, and H is only weakly connected to G^m by the Connectedness Lemma. Now adding back b_{cm} cannot make G strongly connected since $m, c \notin H$ and $m, c \in G^m = G^c$. That is, H is not strongly connected before the unblock step, which contradicts the Connectedness Lemma. □

Proof of Lemma 2 (Chain). By Lemma 5, $G^m \neq G^c$. First, there should be a component path $G^m \to G^c$. Suppose there is no such path. Then adding back b_{cm} cannot make G strongly connected, a contradiction to the Connectedness Lemma. Note that there is no component path $G^c \to G^m$ because if it exists, then together with $G^m \to G^c$, G^m and G^c are strongly connected, a contradiction to $G^m \neq G^c$.

Now I will prove any fixed path (due to the possibility of multiple paths) $G^m \to G^c$ is a clean component chain. Suppose there exists a strong component H such that there exists a component path from H to any component in the path $G^m \to G^c$ (or in reverse direction). Then adding back b_{cm} makes all strong components except H in the path $G^m \to G^c$ strongly connected (unless H is in a different path $G^m \to G^c$ when multiple paths exist), a contradiction to the Connectedness Lemma. Thus, it is a clean component chain.

The possibility of the existence of multiple component chains is shown in Example 3-(3). Also, the existence of strong component that is not even weakly connected to any chain also contradicts the Connectedness Lemma. Thus, G is a connected component chain. □

MSP is pairwise stable, but the following more general result holds.

Lemma 6 (Generalized Pairwise Stability (GPS))**.** *During any step at any price q in the MSP, there exist no two bidders i and j such that $i \in G$ and $j \notin G$, but $b_{ij} > q$.*

Proof. Suppose $b_{ij} > q_j^{II}$. At the beginning of the drop step, there are two cases: (1) $i \in G^j$. Since $G^i = G^j$, i drops together with j; thus, $i \in G$ but $j \notin G$ is impossible; (2) $i \notin G^j$. If $i \notin G$, then the proof is complete. If $i \in G$, then due to $b_{ij} > q_j^{II}$, there exists $G^i \twoheadrightarrow G^j$ by the Chain Lemma, and G^i should drop together with (*earlier than* in φ^{II}) G^j. In any case, at the end of the drop step, both i and j have dropped; thus, after the drop step, $i, j \notin G$. Before the drop step, $j \in G$. □

That is, if j dropped at some q despite $b_{ij} > q$ for some i, then every such i has already dropped earlier than or together with j. When $i = w$ and $q = p$, the GPS Lemma implies PS.

Corollary 8. *The MSP is pairwise stable.*

The following lemma is useful to prove the main theorems.

Lemma 7 (Blocking)**.** *There exists* $b_{ij} \geq q_i^{II}$ *for some bidder* j *(the strict inequality holds if there is no tie in* B*). Therefore,* i *cannot win at some price* $p_i < q_i^{II}$ *in any mechanism that is loser's payment-free and pairwise stable (the weak inequality holds if there is no tie).*

Proof. By the Chain Lemma, in the beginning of the drop step, there exists strong components G^i and H such that $G^i \twoheadrightarrow H$. If $|G^i| = 1$, then $j \in H$ because $G_1 \twoheadrightarrow G_2$ at q implies there exists $b_{ij} \geq q$ such that $i \in G_1$ and $j \in G_2$. If $|G^i| > 1$, then $j \in G^i$ due to the strong connectedness of G^i. Note that such a j can also exist in H. □

It can be easily shown that–if MSP is substituted by φ^I in the statement of each lemma–the GPS Lemma holds for φ^I, but the Blocking Lemma does not hold for φ^I.

D.3 Proof of Theorem 1 and Corollary 6

Proof of Theorem 1. (i) (LPF) By Step (7), only the winner needs to pay.

(ii) (PS) By the GPS Lemma (Corollary 8).

(iii) (weak IR) For any loser l, $b_{wl} \leq \max_{i \neq l} \{b_{il}\}$ by LPF. For the winner, at least one bid b_{iw} for some $i \neq w$ had not been unblocked until the last unblock step. Thus, $p \leq \max_{i \neq w} b_{iw}$.

(iv) (GWRF) By Corollary 3.

(v) (CORF) By Theorem 2-(i), no (anybid) overpay regret.

Uniqueness Without loss of generality, assume there is no tie in B, i.e., any tie in B is broken by a tie-breaking rule in advance. Abusing the notation, "$\varphi(B) = (w, p)$" is used instead of "$\varphi(B) = (w, \mathbf{p})$" if φ is LPF, i.e., $p = p_w$. Let $\varphi(B) = (w, p)$ be the outcome of MSP (denoted by φ), and let φ' be a different mechanism. Then, $\varphi'(B)$ can be one of the following four cases.

(1) $(w' \neq w, p' > p)$: Since w' lost in φ, $q_{w'}^{II} \leq p$ and $w \notin G^{w'}$; otherwise, w cannot win at p in φ. Thus, if w' wins at $p' > p$ in φ', then $V\left(G^{w'}\right)$ has group winner regret at $\underline{q_{w'}^{II}}$ with $q_{w'}^{II} \leq p < p'$ in φ' by Proposition 5.

(2) $(w' \neq w, p' \leq p)$: By the Blocking Lemma (Lemma 7), w' cannot win at $p' \leq q_{w'}^{II}$ in φ' since φ' is LPF and PS; thus, $p' > q_{w'}^{II}$. Then $V\left(G^{w'}\right)$ has group winner regret at $\underline{q_{w'}^{II}}$ with $q_{w'}^{II} < p'$ in φ' by Proposition 5. Note that whether $p' \leq p$ is not used. Thus, this also proves case (1); however, a different proof without using the Blocking Lemma is provided there.

(3) $(w, p' > p)$: Note that this does not immediately imply capped-bid overpay regret in φ' since it is a different mechanism. Let $\mathbf{b}_w^{\overline{q}}$ be a q-capped bid for some q with $p < q < p'$, and let $B' = \left(\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w}\right)$. If w can still win by $\mathbf{b}_w^{\overline{q}}$ in φ', i.e., $\varphi'(B') = (w, p'')$ for some p'',

then w has capped-bid overpay regret in φ' since $p'' \leq q < p'$ by weak IR. Now suppose not, i.e., $\varphi'(B') = (w', p'')$ for $w' \neq w$ and some p'' (note that there is no reason that $p'' \geq q$ yet, which will not matter anyway). However, $\varphi(B') = (w, p)$ since $q > p$ and any $b_{iw} > p$ had never been unblocked. Thus, $\varphi'(B') = (w', p'')$ for any p'' is impossible by cases (1) and (2).

(4) $(w, p' < p)$: $b_{wh} = p > p'$ contradicts the pairwise stability of φ', where h is the threshold bidder.

This completes the proof of Theorem 1, but I also show the independence of the axioms below.

Independence of the axioms Let φ be MSP. In each case, the existence of a new mechanism φ' will be shown by its outcome $\varphi'(B) = (w', \mathbf{p}')$. Before I show each counterexample, note first that if $w' = w$, GWR cannot exist by Proposition 5. Thus, GWRF holds for all cases except for (iv).

(i) (necessity of LPF) $w' = w$, $p'_w = p - \sum_{l \neq w} p'_l$, and $p'_l = \max_{i \neq l} \{b_{il}\}$ for all $l \neq w$.

φ' has a loser's payment when $\max_{i \neq l} b_{il} > 0$ for some $l \neq w$. However, φ' satisfies all other axioms: PS holds since $p = p'$; i.e., the two revenues are the same; weak IR clearly holds; and CORF holds since p'_w is independent of $\mathbf{b}_w$ if w still wins.

(ii) (necessity of PS) $w' = w$, $p'_w = \max_{j \neq h} b_{wj}$, and $p'_l = 0$ for all $l \neq w$. That is, the new winning price p'_w is the second highest b_{wj} for all $j \neq w$, denoted by $b_{wh'}$, and $p'_w < p_w$ is possible when $b_{wh'} < b_{wh}$.

φ' is not PS by $b_{wh} > b_{wh'}$. However, φ' satisfies all other axioms: LPF and weak IR clearly hold; and CORF holds, as in (i).

(iii) (necessity of weak IR) $w' = w$, $p'_w = \bar{b} > p$ (this is possible when $p = b_{wh} < \bar{b}$), and $p'_l = 0$ for all $l \neq w$.

φ' is not weak IR for w when $\max_{i \neq w} \{b_{iw}\} < \bar{b}$. However, φ' satisfies all other axioms: LPF clearly holds; PS holds since $p'_w > p$, i.e., "harder to block"; and CORF holds, as in (i).

(iv) (necessity of GWRF) $\varphi' \equiv \varphi^I$ in Algorithm 1.

φ' is not group winner regret-free, but satisfies all other axioms (Proposition 4).

(v) (necessity of CORF) $w' = w$, $p'_w = \max_{i \neq w} \{b_{iw}\}$, $p'_l = 0$ for all $l \neq w$, and w wins in φ' by any q-capped bid, where $p < q < p'_w$, assuming that $p'_w > p$ (this is possible when $\max_{i \neq w} \{b_{iw}\} > p$).

φ' is not CORF since $p'_w \left(\mathbf{b}^{\bar{q}}_w, \mathbf{b}_{-w}\right) = \max_{i \neq w} \left\{b^{\bar{q}}_{iw}\right\} = q < p'_w$ for q such that $p < q < p'_w$. However, φ' satisfies all other axioms: LPF clearly holds; PS holds, as in (iii); and weak IR holds since $p'_w \leq \max_{i \neq w} \{b_{iw}\}$. □

Proof of Corollary 6. CORF was not used in (2) and (4) of the uniqueness proof. □

D.4 Proof of Theorem 2

(i) Since $p = b_{wh}$, where h is the threshold bidder, winning at $p' < p$ is impossible by pairwise stability. Winning at $p' > p$ by some $\mathbf{b}'_w$ is impossible; otherwise, it implies winning at $p < p'$ is possible by $\mathbf{b}_w$ when the true type is $\mathbf{b}'_w$, a contradiction to the first statement.

(ii) By the GPS Lemma, $q_l^{II} \geq b_{wl}$ (otherwise, at q_l^{II}, $w \in G, l \neq G$, but $b_{wl} > q_l^{II}$, which is impossible). Also, by the Blocking Lemma, l cannot win at $p_l < q_l^{II}$. Thus, the minimum price at which l can win is $p_l \geq q_l^{II} \geq b_{wl}$, which is unprofitable.

(iii) Let Δ_j be the maximum gain by a misreport for j. For winner w, once w wins, $p \leq \max_{i \neq w} \{b_{iw}\}$ and p is fixed by (i); thus, $u_w = -p$. If w loses to some $i \neq w$ by a misreport, then $u'_w = -b_{iw}$. Thus, $\Delta_w \leq \max_{i \neq w} \{|p - b_{iw}|\} \leq \max_{i \neq w, j \neq w} \{|b_{iw} - b_{jw}|\}$. For a loser l, by (ii), i.e., no overturn regret, $\Delta_l \leq \max_{i \neq w} \{|b_{wl} - b_{il}|\} \leq \max_{i \neq w, j \neq w} \{|b_{il} - b_{jl}|\}$.

D.5 Proof of Theorem 3

(i) In fact, this is a corollary of Theorem 2-(iii), but here I provide another proof that uses Theorems 2-(i) and 2-(ii). Let $\mathbf{b}_j = (b, b, ..., b) \in \mathcal{B}_j$. Suppose j wins at p by $\mathbf{b}_j$. Then no other winning price is possible by Theorem 2-(i), i.e., one price for one bidder. Also, losing by a misreport is unprofitable since $p \leq b$ by weak IR. Now suppose j loses by $\mathbf{b}_j$. Losing to another bidder by a misreport does not change the payoff. Winning by a misreport is unprofitable by Theorem 2-(ii), i.e., no overturn regret.

(ii) Without loss of generality, $\underline{b} = 0$ is assumed so that the payment of a bidder with the lowest type, i.e., the zero bid $(0, 0, ..., 0)$, is zero in the second-price auction. Let $\mathbf{b}_j = (b_j, b_j, ..., b_j) \in \mathcal{B}_j$ for all j. By (i), MSP is IC for all bidders. The winner for an efficient allocation is $w \in \arg\max_j \{b_j\}$. By PS and weak IR, w wins in MSP, i.e., MSP is efficient when there are no externalities. Also, weak IR becomes the same as IR since for each bidder j, $b_{ij} = b_j$ for all i. Note that IR by ϵ- and δ-effectiveness are the same since MSP is efficient when there are no externalities. If bidder j with the lowest type is a losing bidder, then $p_j = 0$ by LPF. Even if bidder j with the lowest type is the winner (this is possible if every bidder submits the zero bid), then $p_j \leq 0$ by weak IR, and $p_j \geq 0$ by PS; therefore, $p_j = 0$. Then by the Green-Laffont-Holmstrom Theorem, VCG (in particular, the Groves' scheme) is the only mechanism that is efficient, IC, and IR. Therefore, when there are no externalities, MSP is VCG with zero payment for the lowest type bidder, which is the second-price auction.

Alternative proof Without loss of generality, assume that $b_1 \geq b_2 \geq \cdots \geq b_n$ and that ties (if any) are broken in this sequence. Even when ties exist both within a bidder and across bidders, MSP unblocks all bids within a bidder first (step 3), then starts to unblock the bids of other bidders (step 2 if the bids still exist after the drop step). Thus, until unblocking the last b_{ij} for bidder j, i.e., right before dropping bidder j, all remaining bidders are strongly connected. Thus, MSP drops bidders in the sequence of $n, n-1, ..., 1$; therefore, bidder 1 wins at price b_2. Also, both auctions have no loser's payment. Therefore, MSP has the same outcome with the second-price auction up to ties.

D.6 Proof of other lemmas and propositions

Proof of Proposition 3. Let $R^* = V\left(B_{(q_w^I, k_w^I)}\right) \setminus \{w\}$ and $R(q) = V\left(B'_{(q_w^I(B'), k_w^I(B'))}\right) \setminus \{w\}$, where $B' = \left(\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w}\right)$. (necessity) $R(q) = R^*$ is the same for all $q > q_w^I$ since any bid actually capped by $\mathbf{b}_w^{\overline{q}}$, i.e., $b_{iw} > q$, has not been unblocked before w drops at step 5. At the beginning of step 5 (w is about to drop), $b_{iw} \leq q$ for all $i \in R^*$. Thus, w has winner's regret (WR) at q such that $q_w^I < q < p_w$.

(sufficiency) I will show that WR cannot exist if any of $p_w > q_w^I$ and $w \neq \varphi_1^I(B)$ is not true. First, for any $q < q_w^I$, there exists $i \in R(q)$ such that $b_{iw} > q$ since w drops earlier than i. That is, there cannot be WR at any $q < q_w^I$. Thus, if $p_w \leq q_w^I$, WR cannot exist. Second, suppose $w = \varphi_1^I(B)$. For any $q > q_w^I$, WR cannot exist since $R(q) = \emptyset$. For $q = q_w^I$, there are two cases: (1) "non-binding": $R\left(q_w^I\right) = R(q) = \emptyset$ for $q > q_w^I$; and (2) "binding": $R\left(q_w^I\right) \neq R(q)$ for $q > q_w^I$. In this case, there exists $i \in R(q)$ such that $b_{iw} > q$ since w drops earlier than i. □

Proof of Proposition 5. Let $R^* = V\left(B_{(q_w^{II}, k_w^{II})}\right) \setminus S$ and $R(q) = V\left(B'_{(q_w^{II}(B'), k_w^{II}(B'))}\right) \setminus S$, where $B' = \left(\mathbf{b}_S^{\overline{q}}, \mathbf{b}_{-S}\right)$. (necessity) $R(q) = R^*$ is the same for all $q > q_S^{II}$ since any bid actually capped by $\mathbf{b}_S^{\overline{q}}$, i.e., $b_{ij} > q$ for $j \in S$, has not been unblocked before S drops at step 7. At the beginning of step 7 (S is about to drop), $b_{ij} \leq q$ for all $i \in R^*$ and $j \in S$. Also, by strong connectedness of S, for each $j \in S$, there exists $k_j \in S$ with $k_j \neq j$ such that $b_{k_j, j} \geq q$. Thus, S has group winner regret (GWR) at q such tat $q_S^{II} < q < p_w$.

(sufficiency) I will show that GWR cannot exist if any of $S = \left\{j | q_j^{II} = q_w^{II} \text{ and } k_j^{II} = k_w^{II}\right\}$, $p_w > q_S^{II}$, and $w \neq \varphi_1^{II}(B)$ is not true. First, if $S \neq \left\{j | q_j^{II} = q_w^{II} \text{ and } k_j^{II} = k_w^{II}\right\}$, then S is not a node set of a strong component; thus, (ii) of GWR cannot be satisfied. Second, for any $q < q_S^{II}$, there exists $i \in R(q)$ such that $b_{ij} > q$ for some $j \in S$ since S drops earlier than i. That is, there cannot be GWR at any $q < q_w^I$. Thus, if $p_w \leq q_S^{II}$, GWR cannot exist. Third, suppose $w = \varphi_1^{II}(B)$. Note that $S = \{w\}$ since w cannot win in φ^{II} otherwise. For any $q > q_w^{II}$, GWR cannot exist since $R(q) = \emptyset$. For $q = q_w^{II}$, there are two cases: (1) "non-binding": $R\left(q_w^{II}\right) = R(q) = \emptyset$ for $q > q_w^{II}$; and (2) "binding": $R\left(q_w^{II}\right) \neq R(q)$ for $q > q_w^{II}$. In this case, there exists $i \in R(q)$ such that $b_{iw} > q$ since w drops earlier than i. □

Proof of Proposition 6. Suppose not. Then there exists a bidder who drops at a lower price in φ^I than in φ^{II}. Let j be the bidder who drops first at the lowest price among those bidder(s), then $q_j^I < q_j^{II}$. Let $R_I = V\left(B_{\left(q_j^I, k_j^I\right)}\right)$, i.e., the set of remaining bidders right before j drops. Also, let R_{II} be the set of remaining bidders at $\left(q_j^I, \infty\right)$ (see footnote 26) in φ^{II}. Then $R_{II} \subseteq R_I$ (otherwise, it contradicts the fact that j is the first bidder who drops at a lower price in φ^I). However, the fact that j drops at q_j^I in φ^I implies $b_{ij} \leq q_j^I$ for all $i \in R_I \backslash \{j\}$. Thus, $b_{ij} \leq q_j^I$ for all $i \in R_{II} \backslash \{j\}$, which implies j has to drop at q_j^I or a lower price in φ^{II}, i.e., $q_j^{II} \leq q_j^I$, a contradiction to $q_j^I < q_j^{II}$. The possibility of $\mathbf{q}^I \neq \mathbf{q}^{II}$ can be easily shown by an example that is free of winner's regret, but not free of group winner regret, e.g., the motivating example. □

The following lemma is needed for the case $b_{ij} \leq 0$ as explained in footnote 22. The same result holds for all mechanisms in Algorithms 1 to 4.

Lemma 8 (Linearity)**.** *In the MSP, the following holds up to ties:* $x(B) = x(B + c_2) = x(c_1 \cdot B)$*,* $\rho(B + c_2) = \rho(B) + c_2$*, and* $\rho(c_1 \cdot B) = c_1 \cdot \rho(B)$ *for* $c_1 \in \mathbb{R}_{++}$ *and* $c_2 \in \mathbb{R}_+$*.*

Proof. Note that $B' = B + c_2$ should be written as $B' = \left(b'_{ij}\right)$, where $b'_{ij} = b_{ij} + c_2$ if $i \neq j$, otherwise $b'_{ij} = 0$. However, since the diagonal is unused in MSP, $B + c_2$ is used for simplicity. Let $w = x(B)$, $p = \rho(B) = b_{wh}$ for the threshold bidder h, $B' = B + c_2$, and $B'' = c_1 \cdot B$, i.e., $b'_{ij} = b_{ij} + c_2$ and $b''_{ij} = c_1 \cdot b_{ij}$. Then b'_{ij}'s and b''_{ij}'s have the same order of b_{ij}'s since $f(x) = x + c_2$ and $g(x) = c_1 \cdot x$, $x \in \mathbb{R}$, are isotone. Thus, all three auctions with B, B', and B'' have the same winner and threshold bidder up to ties since the sequence of bids that are unblocked in the unblock step and the sequence of bidders that are dropped in the drop step are the same. Therefore, $x(B') = x(B'') = w$, $\rho(B') = b'_{wh} = b_{wh} + c_2$, and $\rho(B'') = b''_{wh} = c_1 \cdot b_{wh}$. □

The following lemma shows that MSP ends in finite time. As mentioned in footnote 37, lemmas 2, 8, and 9 show that MSP is well-defined.

Lemma 9. *The MSP ends in finite time, i.e.,* $|V| = 1$ *happens in finite time.*

Proof. $|V|$ monotonically decreases at each drop step since there is no step where it increases. Suppose $|V|$ stops decreasing at some $k > 1$. However, for G to be strongly connected, $|E| \geq |V|$. Since one b_{ij} is unblocked at each unblock step, i.e., $|E|$ decreases by one, eventually $|E| < |V|$ must happen. Then, $|V|$ has to further decrease in the drop step, a contradiction. $|V| = 0$ is impossible since the drop step always leaves G^c undropped. □

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