The Object Allocation Problem with Random Priorities

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Abstract

The existing priority-based object allocation literature restricts objects' priorities to be deterministic. However, agents might be probabilistically prioritized ex-ante, for instance, through a non-uniform tie-breaking rule. This paper generalizes the deterministic setting by allowing priorities to be random. In this probabilistic environment, we first introduce a fairness notion called *claimwise stability* in the spirit of the usual stability of Gale and Shapley (1962). We show that while the celebrated deferred acceptance mechanism (Gale and Shapley (1962)) is claimwise stable, it might be ordinally dominated by another claimwise stable rule. We then introduce a new mechanism called the *constrained probabilistic serial*, which is built on the probabilistic serial mechanism of Bogomolnaia and Moulin (2001). It is both claimwise stable and constrained ordinally efficient. The paper then systematically compares the deferred acceptance and constrained probabilistic serial mechanisms in terms of the strategic and fairness properties. Lastly, we provide an axiomatic characterization of the constrained probabilistic serial mechanism.

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1 Introduction

In a priority-based object allocation problem, there are sets of agents and indivisible objects that are to be distributed among the agents. Each agent has a strict preference over the objects and is entitled to receive one object. Similarly, each object has a priority order (possibly with ties) over the agents. No monetary transfer is allowed. Student placement in public schools (see Abdulkadiroğlu and Sönmez (2003)) is a well-known real-life example of such allocation problems.

Theoretical advancements in the object allocation literature have proved to be useful in practice. Many student placement systems, including the two largest Boston and New York City school districts in the United States, have been successfully redesigned by economists with the guidance of theory (see Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b)). However, they are still not problem free. For instance, the following passage is from Boston mayor Thomas Menino's 2012 State of the City address:

"*Pick any street. A dozen children probably attend a dozen different schools. Parents might not know each other; children might not play together. They cannot carpool, or study for the same tests. We will not have the schools our kids deserve until we build school communities that serve them well.*"

A striking fact supporting Mayor Menino's remarks is that 19 children from the same street in Boston attend 15 different schools, which are far away from each other (Ebbert and Ulmanu (2011)).[1] As having a community's children attend the same schools is one of the key factors for community cohesion, some have advocated replacing school choice[2] with the neighborhood system.[3] However, school choice definitely adds important value to student

[1]They report that it involves combined travel of 182 miles each day.

[2]In the school choice system, students submit a ranked list of schools in order of preference, and schools have priorities over students. A centralized institution then determines a student-school matching by running a certain matching algorithm. In the United States, many school districts, including Boston, New York, San Francisco, and Chicago, employ the school choice system.

[3]Having the same community children attend different schools has other important undesirable consequences, including high transportation costs. Seelye (2012) reports that transportation costs \$80.4 million a year in Boston, which is about 9.4% of the school system's operating budget, almost twice the national average.

placement in that it enables families to express their preferences and honors them as much as possible, thereby improving educational outcomes.[4] Hence, we should look for ways to improve community cohesion while keeping the school choice system.

To this end, in the following, we propose a way to improve community cohesion under the current school choice system. In reality, school priorities involve large indifference classes; in other words, many students are tied in school priority lists. As the well-known student placement mechanisms in use in various school districts are not well-defined in the presence of such ties, common practice is to first break the ties through a random tie-breaking rule and then to run the placement mechanism based on the obtained strict priorities. This random tie-breaking stage might enable us to improve community cohesion at least in expectation.

In the course of tie-breaking, the central authority can employ a non-uniform random tie-breaking rule favoring students within a given community, which would eventually assign them to the same schools (if they so desire).[5,6] For instance, let us consider three students, i, j, and k, such that the first two students are from the same community. Assume that there are two schools, a and b, with two available seats in the former and one in the latter, and that all the students are tied in both schools' priorities. Let the students unanimously prefer school a to school b. If the ties are broken uniformly (hence, any strict priority order is equally likely), then each student is matched with school a with probability 2/3 under the widely used Gale and Shapley (1962)'s deferred acceptance mechanism (hereafter, DA). Therefore, the students from the same community, i and j, attend the same school with probability 1/3 under the ex-post implementation of the DA. Let us now break the ties non-uniformly so that students i and j are the top two students with probability 2/3 at

[4]For instance, in their book "*Politics, Markets, and America's Schools*", Chubb and Moe (1990) argue that school choice is the most promising reform for improving school quality and thereby improving educational outcomes.

[5]In practice, the uniform tie-breaking rules, under which each of the same priority students has the same chance of having higher priority than the others ex-post, are commonly used. We will return to this issue in detail in Footnote 18.

[6]Ashlagi and Shi (2014) also consider the cohesion problem. They propose another approach to lessen it: instead of the currently proposed solution of non-uniform random tie-breaking, they suggest the correlated ex-post implementation of random assignments.

school a. In this case, they go to the same school with probability 2/3 under the ex-post implementation of the DA, increasing community cohesion in expectation.

A similar problem that can be lessened (in expectation) through non-uniform tie-breaking is having fewer students assigned to walk-zone schools than intended in the Boston public schools. Dur et al. (2013) document that this problem exists even though walk-zone students are given higher priority in half of the school seats in the current system. They offer alternative ways (increasing walk-zone slots and changing the precedence order with respect to which slots are filled) that have been shown to be useful in the data but which might exacerbate the problem in theory.[7] On the other hand, a non-uniform tie-breaking rule favoring walk-zone students would assign more of them to their walk-zone schools (as long as they desire to attend one), hence reducing the problem both in theory and in the data. It therefore can be seen as another possible remedy for the problem. We can indeed increase the number of real-life applications of non-uniform tie-breaking. Generally speaking, however, a non-uniform tie-breaking rule can be used to incorporate various constraints or society's preferences into the matching design.

Once we are willing to employ a non-uniform tie-breaking rule, we need to enrich the traditional priority-based object allocation theory. This is because randomly breaking ties in a non-uniform way means that seemingly equal-priority students in deterministic priorities are not "exactly" tied in the mechanism designer's eyes. Therefore, the present deterministic priority domain of the existing literature is not capable of representing students' exact positions at schools; hence, we need a richer class of priorities. To this end, we allow priorities to be random in this paper, which enables us to fully capture students' positions and incorporate them into the mechanism design. Our approach is important because what the mechanism design recommends can change drastically from deterministic priorities to random ones. For instance, let us consider two agents i and j and one object a. Assume

[7] They also consider two lottery numbers instead of one to break ties and empirically show that it reduces the problem. However, an important drawback of this method is that it generates efficiency losses (Abdulkadiroğlu et al. (2009)).

that both of the agents have the same priority for the object; that is, they are tied.[8] Then, with respect to this deterministic priority order, as the agents are equal, the "fair" matching is random, giving the object to each agent with probability 1/2. In contrast, let us consider the situation in which one agent, say agent i, is favored in the course of random tie-breaking with a 2/3 chance of having higher priority. This means that the probability that agent i has higher priority than agent j is 2/3 with respect to the random priority order. That is, as opposed to what the above deterministic priority states, the agents are not exactly tied. Then, if we take the random priority as a primitive of the problem, since the agents are no longer equal, giving 1/2 of the object to each agent would not be fair.[9]

The current paper studies the object allocation problem with random priorities; to the best of our knowledge, it is the first paper in this direction. We first introduce a fairness notion called "claimwise stability" in a setting in which objects' priority order profile is a lottery over deterministic and strict priorities, and assignments (matchings) are allowed to be random. We can consider objects as divisible and the entries of an assignment as the objects' assigned shares to agents. Then, for a given problem and matching, we say that agent i has a justified claim against agent j for object a if the assigned share of object a to the latter is greater than the probability that he has higher priority than the former at object a[10] plus agent i's total share of his more preferred objects. We say that a matching is claimwise stable if no agent has a justified claim.

What is the intuition behind claimwise stability? The two main motivating arguments stem from the claim problem (a.k.a. bankruptcy or rationing) and the matching literature. For ease of understanding, let us consider a problem consisting of two agents, i and j, and one object a. We can interpret the probability of agent i having higher priority than agent j

[8] While we motivate the research in the school choice context, it is not restricted to this context. Because of this, we will use agent-object rather than student-school terminology for general descriptions.

[9] We will introduce a fairness notion in the random priority setting. In this particular example, it will give the object to agent i with probability 2/3.

[10] The probability that agent j has higher priority than agent i at object a is defined as the sum of the probabilities of the deterministic priorities (with respect to the given lottery) under which the former has higher priority than the latter at the object.

as the share of object a for which the former has claim against the latter (similarly, for agent j). Then, claimwise stability recommends giving each agent as much share as his claim. That is, it advocates distributing the shares proportionally to agents' claims. This kind of claim-proportional allocation rule is very common in practice, and it is often considered as a fairness criterion in claim problems (see, for example, Thomson (2003)). On the other hand, in our general setting with more than one object, agent i (similarly for agent j) can satisfy some fraction of his demand with his more preferred objects. In this case, naturally, claimwise stability allows agent j to acquire more of object a as much as up to that fraction.

Another motivating argument for claimwise stability stems from the stability of Gale and Shapley (1962) (hereafter, we often refer to the stability of Gale and Shapley (1962) as *usual stability* to avoid confusion). To this end, we use the "consumption process" representation of random assignments by Bogomolnaia and Heo (2012).[11] Any assignment can be seen as the outcome of a consumption process where, over the unit time interval, agents continuously consume objects at the speed of one in decreasing order of their preferences.[12] In the course of this consumption process, claimwise stability rules out any time instant at which an agent envies someone else for the object he is consuming while the former has higher priority.[13] In other words, it requires the usual stability at every point of time in the course of the consumption process. Hence, it can be seen as a natural generalization of the usual stability to the current random environment. Claimwise stability indeed collapses to the usual stability in the conventional deterministic priority domain for deterministic matchings.[14]

Claimwise stability is the central notion throughout this paper. We then look for mechanisms satisfying it along with other desirable properties. The most natural mechanism to consider is the DA because of its superior properties in the conventional setting and its

[11]This representation was first introduced by Heo (2010) and called "the preference-decreasing consumption schedule."

[12]Each different assignment corresponds to a different stopping time (i.e., the time instances at which an agent stops consuming an object) configuration in the consumption process.

[13]This point is explained in detail in the model section.

[14]Its relations to other probabilistic fairness notions are discussed in the model section.

wide usage in real-life problems.[15] While the DA turns out to be claimwise stable, its main drawback is that agents might prefer another claimwise stable assignment for every cardinal payoff profile compatible with their ordinal rankings. That is, formally speaking, the DA might be "*ordinally dominated*" by another claimwise stable mechanism.

This important handicap of the DA leads us to look for other claimwise stable solutions. For this purpose, we introduce a "*constrained probabilistic serial mechanism*" (henceforth, CPS), which is built on the probabilistic serial mechanism (hereafter, PS) of Bogomolnaia and Moulin (2001). In their setting, agents are not prioritized (in other words, they are equal), and in the course of the PS, they continuously acquire the probability shares of objects (in decreasing order of their preferences) until the objects are totally exhausted. As agents are treated equally in acquiring objects' shares in the course of the PS, claimwise stability is not satisfied in the current setting with priorities. To fix it in the CPS, we allow agents to acquire the shares of objects until either relevant claimwise stability constraints start binding or the objects are totally exhausted (whichever occurs first). Except for this, the CPS works the same as the PS. Fortunately, this easy modification makes the CPS claimwise stable.

In contrast to the DA, the CPS is not ordinally dominated by another claimwise stable mechanism (it might be ordinally dominated by a non-claimwise stable mechanism; nevertheless, it is not a problem specific to the CPS because claimwise stability and ordinal efficiency are incompatible). The CPS coincides separately with the DA and the PS in their respective domains in which they admit superior properties. We then systematically compare the DA and the CPS in terms of the strategic and fairness properties. While the CPS turns out to be constrained ordinally efficient (i.e., not ordinally dominated by another claimwise stable mechanism) and performs better than the DA in terms of fairness, the converse is true with respect to strategic issues. Lastly, we provide an axiomatic

[15] The New York City (which has the largest public school system in the country, with over a million students) and Boston (which has over $60,000$ students enrolled in the public school system) school districts have been using the DA (Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b)) to assign students to schools.

characterization of the CPS: it is the unique non-wasteful, claimwise stable, and binding mechanism. Non-wastefulness is a desirable standard efficiency property. Bindingness, on the other hand, requires each agent to continue consuming his preferred objects as long as no one else causes him to stop through claimwise stability constraints.

2 Related Literature

This paper is broadly related to the probabilistic matching literature. Vate (1989) and Rothblum (1992) characterize ex-post stable probabilistic ("fractional" in their paper) matchings.[16] Manjunath (2014) defines a competitive equilibrium notion in a setting in which agents can be matched in fractions and shows its existence. Hylland and Zeckhauser (1979) introduce a market procedure that achieves envy-freeness and efficiency (with respect to Von Neumann-Morgenstern utilities), yet it is not strategy-proof.[17] Baiou and Balinski (2002) and Alkan and Gale (2003) study a "schedule matching" problem in which each worker's total working hours are allocated to firms in fractions.

The closest work to the current paper is Kesten and Ünver (2014). They introduce two different ex-ante stability notions for random assignments in the school choice context with deterministic and coarse priorities. They show that the DA does not satisfy both of them and introduce two mechanisms satisfying their stability notions along with some other desirable properties. While the priority structures of Kesten and Ünver (2014) and the current paper seem different at first glance, they are not. We can define a natural isomorphism between them and show that our model subsumes theirs (yet the respective notions, mechanisms, and results are independent). A detailed formal discussion on the relation between the papers is provided in Section 3.

[16] Roth et al. (1993) and Teo and Sethuraman (1998) provide alternative proofs of this characterization. The former also shows that the lattice structure of the stable matchings extends to the ex-post stable matchings with respect a partial ordering involving stochastic dominance.

[17] Zhou (1990) proves that efficiency, strategy-proofness, and symmetricity (a weaker property than envy-freeness) are incompatible in economies with at least three agents.

As mentioned in the introduction, non-uniform tie-breaking has important applications, and a variety of its other aspects have been studied. Erdil and Ergin (2008) demonstrate that depending on the means of breaking ties, the DA might be dominated by another stable matching. In the same context, Abdulkadiroğlu et al. (2009) compare the single tie-breaking and multiple tie-breaking rules[18] under the DA in terms of the efficiency and incentive properties. Their work reveals that more students receive their top choices under the single tie-breaking rule (yet there is no stochastic dominance relation between them); hence, they conclude that it has better efficiency properties than the multiple tie-breaking rule.

Another important related paper is Bogomolnaia and Moulin (2001). They introduce the PS for the priority-free object allocation problem and show that it is ordinally efficient, whereas the well-known random priority mechanism[19] is not. In the current paper, we adopt the PS to a priority-based object allocation setting for the first time and show that the CPS is constrained ordinally efficient, whereas the well-known DA is not. The results show that the appealing properties of the DA do not carry over to the current random environment. Similar points are found in various other papers in different settings, including Erdil and Ergin (2008) and Kesten and Ünver (2014). In an incomplete information setting regarding the preferences of students, Featherstone and Niederle (2008) find that truth-telling can be an equilibrium under the Boston mechanism and can first-order stochastically dominate the DA. Similarly, in an incomplete information setting with common ordinal preferences with no priority, Abdulkadiroğlu et al. (2011) demonstrate that every student at least weakly prefers any symmetric equilibrium of the Boston mechanism to the dominant strategy equilibrium of the DA. Similar results are obtained in Troyan (2012) and Miralles (2008).

[18] In the single tie-breaking rule, each student is randomly given a single number, which is used to break the ties in every school. On the other hand, in the multiple tie-breaking rule, each student is randomly given a (possibly different) number for each different school, and the ties at a school are broken depending on the given numbers for that school.

[19] Under the random priority mechanism, an ordering of agents is chosen uniformly; the agents then choose their remaining favorite object one at a time in order.

The PS has received a great deal of attention in the literature. It has been extended to various richer domains, in part because of its superior properties. Katta and Sethuraman (2006) generalize the PS to the weak preference domain. Heo (2014) considers the more general multi-unit demand case and extends the PS to that setting. Yilmaz (2010, 2009) lets agents have private endowments in the strict and weak preference domains and generalizes the PS to those domains. Kojima (2009) extends the PS to the multi-quota setting. Aliogullari et al. (2013) modify the PS to increase the expected number of agents matched with their more preferred objects. Budish et al. (2013) introduce a variant of the PS accommodating various real-life constraints. In addition to this line of research, there has been a recent surge in characterizations of the PS. Bogomolnaia and Heo (2012) and Hashimoto et al. (2014) provide axiomatic characterizations of the PS.

3 The Model and Results

There are finite sets of agents N and objects O that are to be distributed among the agents. Each agent $i \in N$ has a *preference relation* R_i, which is a complete, transitive, and antisymmetric binary relation over O and being unassigned, denoted by $\emptyset$ (the null object). We write aP_ib whenever aR_ib and $a \neq b$. Object $a \in O$ is *acceptable* to agent i if $aP_i\emptyset$; it is otherwise *unacceptable*. Let $\mathcal{R}$ and q_a be the set of preference relations and the number of copies of object a, respectively. The null object is not scarce; that is, $q_\emptyset = \infty$. For ease of exposition, we present the results for the case of no null object, $q_a = 1$ for each $a \in O$, and $|N| = |O|$. However, in Section 4, we show that all the results carry over to general cases.

In the conventional priority-based object assignment model, each object a is endowed with a deterministic priority order $\succ_a$, which is a complete, strict, and transitive binary relation over N. Let $\succ = (\succ)_{a \in O}$ denote a deterministic priority order profile and let ζ be the set of such profiles.

This paper departs from the above conventional setting and allows priorities to be random. Formally, we write Δ for the (random) priority order profile of the objects, which is a probability distribution (lottery) over ζ. We write Δ_a for the priority order of object a, which is the marginal probability distribution of the priority order of object a under Δ. There is no restriction on Δ; hence, objects' priorities may be independent as well as correlated.[20] Let $\Delta(\succ)$ be the probability of $\succ = (\succ_a)_{a \in O}$ under Δ. We write $supp(\Delta) = \{\succ \in \zeta : \ \Delta(\succ) > 0\}$ for the support of Δ. We define $Pr_\Delta(i \succ_a j) = \sum_{\succ \in \zeta: \ i \succ_a j} \Delta(\succ)$. In words, it is the probability that agent i has higher priority than agent j at object a. In the rest of the paper, we fix the set of agents and objects and simply write (R, Δ) for the problem.

A *matching* $\sigma = [\sigma_{i,a}]_{i \in N, a \in O}$ is a matrix such that for all $i \in N$ and $a \in O$, (i) $0 \leq \sigma_{i,a} \leq 1$, (ii) $\sum_{a \in O} \sigma_{i,a} = 1$, and (iii) $\sum_{i \in N} \sigma_{i,a} = 1$. Here, $\sigma_{i,a}$ represents the probability that agent i is matched with object a. Let σ_i and σ^a denote the random assignments of agent i and object a at σ, respectively. A matching σ is *deterministic* if $\sigma_{i,a} \in \{0, 1\}$ for all $i \in N$ and $a \in O$. Let $\mathcal{X}$ be the set of all matchings. We write $\mathcal{M}$ for the proper subset of $\mathcal{X}$ consisting of deterministic matchings.

A probability distribution λ over $\mathcal{M}$ is called a lottery. Formally, $\lambda = (\lambda_\mu)_{\mu \in \mathcal{M}}$ is such that for all $\mu \in \mathcal{M}$, $0 \leq \lambda_\mu \leq 1$ and $\sum_{\mu \in \mathcal{M}} \lambda_\mu = 1$. We write σ^λ for the matching induced by λ, that is, $\sigma^\lambda_{i,a} = \sum_{\mu \in \mathcal{M}: \ \mu_i = a} \lambda_\mu$.

Fact 1 (Birkoff-Von Neumann)**.** *Any matching can be induced by a lottery λ over $\mathcal{M}$.*[21]

Because of the above well-known fact, in the rest of the paper, we consider matchings instead of lotteries. Given an agent i and an object a, let $SU(R_i, a) = \{c \in O : \ cP_i a\}$ (the strict upper contour set of agent i at object a), and $U(R_i, a) = SU(R_i, a) \cup \{a\}$ (the upper contour set of agent i at object a).

[20] This generality is important in both theory and practice. The single and multiple tie-breaking rules (described in Footnote 18), which are well-studied and well-used, yield independent and correlated priorities, respectively (Abdulkadiroğlu et al. (2009)).

[21] This fact is extended to the multi-quota case by Kojima and Manea (2010).

Given two assignments σ_i and σ'_i of agent i, the former *ordinally dominates* the latter if $\sum_{c \in U(R_i,a)} \sigma_{i,c} \geq \sum_{c \in U(R_i,a)} \sigma'_{i,c}$ for each $a \in O$, with holding strictly for at least one object. That is, σ_i *first-order stochastically dominates* σ'_i with respect to R_i. A matching σ ordinally dominates σ' if for all $i \in N$, either σ_i ordinally dominates σ'_i or $\sigma_i = \sigma'_i$, with the former holding for at least one agent. A matching is *ordinally efficient* if it is not ordinally dominated by another matching.

In the conventional setting with deterministic priorities, Gale and Shapley (1962) say that a deterministic matching μ is *stable* at $(R, \succ)$ if there exists no pair of agents i, j such that $\mu_j P_i \mu_i$ and $i \succ_{\mu_j} j$. Since its introduction, stability has been one of the major desiderata in both theoretical and practical matching design problems. However, it becomes silent in the current domain whenever priorities or matchings are random. Therefore, we first introduce the following natural generalization of stability, which will be the main notion in the rest of the paper.

Definition 1.

(i) Given a problem (R, Δ) and matching σ, we say that agent i has a justified claim against agent j for object a if $\sigma_{j,a} > Pr_\Delta(j \succ_a i) + \sum_{c \in SU(R_i,a)} \sigma_{i,c}$.

(ii) A matching σ is claimwise stable if no agent has a justified claim against someone else for an object.

Remark 1. If object a is the top choice of agent i, then $SU(R_i, a) = \emptyset$. In this case, $\sum_{c \in SU(R_i,a)} \sigma_{i,c} = 0$ for any matching σ.

Remark 2. It is clear from the definition that in the conventional setting with deterministic priorities, claimwise stability collapses to the usual stability for deterministic matchings.

We can motivate the claimwise stability notion through the proportional distribution criterion in the claim problem (a.k.a. bankruptcy or rationing) literature and the usual stability of Gale and Shapley (1962). For any given pair of agents i and j and object a, we can interpret $Pr_\Delta(j \succ_a i)$ as the claim of agent j on object a against agent i. If, for the moment, we assume that object a is the only object, then claimwise stability recommends distributing the assignment probabilities proportionally to the agents' claims. This kind of assignment rule is commonly used and is often considered as a fairness criterion in claim problems (for instance, see Thomson (2003) and Moulin (2002)). On the other hand, in the case of multiple objects, claimwise stability naturally lets agent j obtain more of object a as much as up to $\sum_{c \in SU(R_i,a)} \sigma_{i,c}$ (the fraction of agent i's demand satisfied with his more preferred objects). Hence, claimwise stability can be seen as a natural counterpart of the proportional allocation principle in the current object allocation model.

To explore the connection between the claimwise and usual stability notions, we first invoke the consumption process representation of random assignments by Bogomolnaia and Heo (2012). Any random assignment can be seen as the outcome of a consumption process where, over the unit time interval, agents continuously acquire the probability shares of objects in decreasing order of their preferences at the speed of one. The assigned probabilities under a random assignment represent the time-shares in which agents consume the corresponding objects in the course of the consumption process. Let us now consider a pair of agents i and j, an object a, and a matching σ. We can consider $Pr_\Delta(j \succ_a i)$ as the time-share in which agent j has higher priority than agent i for object a. Similarly, let $\sum_{c \in SU(R_i,a)} \sigma_{i,c}$ represent the time-share in which agent i consumes his more preferred objects to object a at σ. Then, if $\sigma_{j,a} > Pr_\Delta(j \succ_a i) + \sum_{c \in SU(R_i,a)} \sigma_{i,c}$, it implies that there is a time interval during which agent j consumes object a while agent i consumes one of his less preferred objects even though he has higher priority for object a (see Figure 1 below). Therefore, claimwise stability rules out any time instant at which an agent envies someone else for the object he is consuming while the former has higher priority. In other words, it requires the usual stability at every point of time in the course of the consumption process.

As already pointed out in Remark 2, in the conventional setting with deterministic priorities, claimwise stability coincides with the usual stability for deterministic matchings.

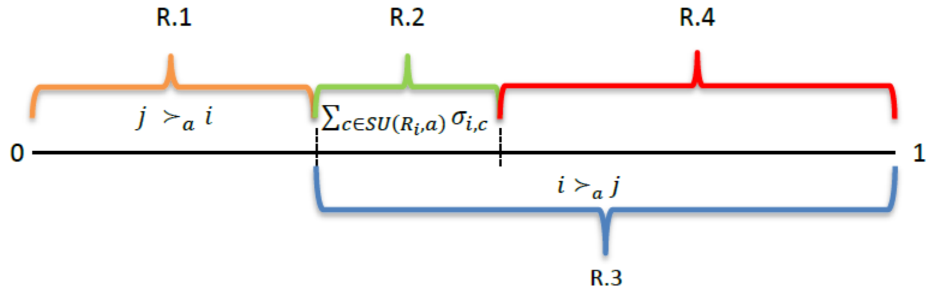

Figure 1: Agents consume objects over the above unit time interval. The orange region (R.1) represents the time-share in which agent j has higher priority than agent i for object a (its length is $Pr_\Delta(j \succ_a i)$). Similarly, agent i is prioritized in the blue region (R.3). On the other hand, the green region (R.2) shows the time-share in which agent i consumes his more preferred objects. Therefore, if $\sigma_{j,a} > Pr_\Delta(j \succ_a i) + \sum_{c \in SU(R_i,a)} \sigma_{i,c}$ (the sum of the orange (R.1) and green regions (R.2)), it implies that agent j consumes object a for some period of time in the red region (R.4). However, this means that for some period of time, agent j consumes object a while agent i consumes a less preferred object even though he has higher priority at object a. In other words, the usual stability is violated within that time interval (note that this is independent of the region locations on the unit interval above).

However, there are non-deterministic claimwise stable matchings as well.[22] Since stability is vital for the well-working of markets, it is important for a (random) matching to implement stable matchings ex-post. For this purpose, a matching σ is said to be *ex-post stable* at a problem with deterministic priorities $(R, \succ)$ if it can be written as a lottery over stable matchings. The question follows of whether or not claimwise stable matchings are ex-post stable. Below, we show that the answer is affirmative.

Proposition 1. *In the deterministic priorities domain, any claimwise stable matching is ex-post stable.*

Proof. See the Appendix. □

Remark 3. It is easy to verify that in the deterministic priorities domain, any lottery over stable matchings is claimwise stable. This fact, along with the above proposition, shows that the sets of claimwise stable and ex-post stable matchings coincide with each other in the conventional setting with deterministic priorities.

A *mechanism* ψ is a function that assigns a matching for every problem. A mechanism ψ is claimwise stable if $\psi(R, \Delta)$ is claimwise stable at any problem (R, Δ).

[22]For instance, in the conventional deterministic priorities domain, it is easy to verify that any lottery over stable matchings is claimwise stable.

Kesten and Ünver (2014) introduce two stability notions for random assignments when objects have deterministic and coarse priorities. Let us write $i \sim_a j$ to denote that agents i and j have the same priority at object a. Kesten and Ünver (2014) say that matching σ induces ex-ante schoolwise justified envy of student i toward student j if $i \succ_a j$, $\sigma_{j,a} > 0$, and $\sigma_{i,b} > 0$ for some school b such that aP_ib. They call matching σ ex-ante stable if it does not cause any ex-ante schoolwise justified envy. Moreover, they say that matching σ induces ex-ante schoolwise discrimination between students i and j if $i \sim_a j$, $\sigma_{i,a} < \sigma_{j,a}$, and $\sigma_{i,b} > 0$ for some school b where aP_ib. Matching σ is called strongly ex-ante stable if it eliminates both ex-ante schoolwise justified envy and ex-ante schoolwise discrimination.

The relation between Kesten and Ünver (2014) and the current paper hinges on the priority structures. We can replicate their priorities in our setting via a natural isomorphism; hence, our model subsumes theirs. Namely, for any given Kesten and Ünver (2014)'s problem $(R, \succeq)$,[23] define the counterpart problem (R, Δ) of it in the current setting as follows:

$$\text{Let } \zeta^* = \{\succ'= (\succ'_a)_{a\in O} \in \zeta : \; i \succ'_a j \;\Leftrightarrow\; i \succ_a j\}, \text{ and}$$
$$\text{for any } \succ', \succ'' \in \zeta^*,\ \Delta(\succ') = \Delta(\succ'').$$ [24]

That is, under the above Δ, we have

$$(i)\ i \sim_a j \Leftrightarrow Pr_\Delta(i \succ_a j) = 1/2, \text{ and for any } k \in N \setminus \{i, j\},\ Pr_\Delta(i \succ_a k) = Pr_\Delta(j \succ_a k),$$
$$(ii)\ i \succ_a j \Leftrightarrow Pr_\Delta(i \succ_a j) = 1.$$

In words, any equal-priority students at any object in Kesten and Ünver (2014) are also equal at the same object at Δ, and any higher priority student continues to be prioritized at Δ as well. Hence, we completely replicate their priorities in our setting. The converse, however, is not true.[25] Hence, our model is more general than theirs.[26]

[23] They denote the deterministic and coarse priority profile of objects by $\succeq$.

[24] That is, Δ is the uniform distribution with the support of ζ^*.

[25] For instance, if $Pr_\Delta(i \succ_a j) = 1/3$, then students i and j do not have the same priority, nor does either of them have higher priority.

[26] Apart from this theoretical generality, our paper also has practical appeal because of its practical applications, as mentioned in the Introduction.

While the current model is more general than that of Kesten and Ünver (2014), there is no direct relation between the stability notions or between the mechanisms. First, as their model is narrower, their stability notions become silent in a very large subdomain of our setting. However, if we consider the above restricted domain (that is, the set of uniform distributions over different ζ^* defined as above), then we can easily observe that strong ex-ante stability implies claimwise stability, yet the converse is not true.[27] As the stability notions are different, the mechanisms handle different constraints, making them independent as well. We will discuss the relation between the mechanisms in more detail later.

The next natural research direction is to construct a mechanism that produces a claimwise stable matching and admits other desirable properties (if possible) as well. For this purpose, Gale and Shapley (1962)'s celebrated DA is the first natural candidate because of its superior properties in the conventional deterministic priorities domain.[28]

3.1 The Deferred Acceptance Mechanism (DA)

Below, we outline the DA.

Step 1. Each agent applies to his first choice object. Each object that receives one or more applications is tentatively assigned to the highest priority applicant, and the other applicants are rejected.

In general,

Step t. Each agent who was rejected in step $(t-1)$ applies to his next object. Each object is tentatively assigned to the highest priority agent among the current applicants and the tentatively assigned one in the previous step (if any), and the other applicants are rejected.

[27] Any lottery over stable matchings is claimwise stable; however, it is not necessarily ex-ante stable. On the other hand, an ex-ante stable matching may not be claimwise stable either.

[28] The DA dominates any other stable matching in terms of efficiency. Moreover, it is the unique strategy-proof stable mechanism.

The algorithm terminates whenever every agent is tentatively assigned, and the tentatively held applications at the termination step are realized as the final assignments.

The DA is defined for strict priorities. In other words, it is not well defined for random priorities. A natural generalization of the DA to the current setting is to apply the DA to each strict priority profile in the support of Δ and to find their weighted sum, where the weights come from the probabilities of the associated strict priority profiles under Δ. Formally speaking, for a given problem (R, Δ), $DA(R, \Delta)_{i,a} = \sum_{\succ \in supp(\Delta)} \Delta(\succ) DA(R, \succ)_{i,a}$ for each $i \in N$ and $a \in O$.

From Gale and Shapley (1962), we know that the DA is stable in the conventional deterministic priorities domain. The following proposition shows that it is claimwise stable in our general random priorities setting as well.

Proposition 2. *The DA is claimwise stable.*

Proof. We want to show that $DA(R, \Delta)$ is claimwise stable at any problem (R, Δ). By the definition of the DA, we have $DA(R, \Delta) = \sum_{\succ \in supp(\Delta)} \Delta(\succ) DA(R, \succ)$. As $DA(R, \succ)$ is stable at any $\succ$, an agent is either the top priority one for his matched object or any other higher priority agent is matched with one of his more preferred objects. This basically implies that for any agents i and j and object a, $DA(R, \Delta)_{i,a} \leq Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j, a)} DA_{j,c}$, which shows that the DA is claimwise stable. □

Given that we now have a claimwise stable mechanism, a next desideratum is to identify the most efficient ones. Mechanism ψ is *ordinally efficient* if $\psi(R, \Delta)$ is not ordinally dominated at any (R, Δ).

It is well known that the usual stability and efficiency are incompatible even in the conventional deterministic setting (Balinski and Sönmez (1999)). On the other hand, from Proposition 1, we know that any claimwise stable matching can be written as a lottery over stable matchings. These results imply that claimwise stability and ordinal efficiency are incompatible.

Corollary 1. *Suppose* $|N| \geq 3$.[29] *Then, there is no claimwise stable mechanism that is ordinally efficient.*

Given the above negative result, we next look for the claimwise stable mechanisms that are efficient within the class of claimwise stable solutions. We say that a matching σ is *constrained ordinally efficient* if no other claimwise stable matching ordinally dominates it. Mechanism ψ is constrained ordinally efficient if $\psi(R, \Delta)$ is constrained ordinally efficient at every problem (R, Δ).

It is well known that the DA dominates (in terms of efficiency) any other stable mechanism in the conventional deterministic domain (Gale and Shapley (1962)). In sharp contrast to this fact, the following result demonstrates that it is not even constrained ordinally efficient in the current framework.[30]

Proposition 3. *Suppose* $|N| \geq 4$. *Then, the* DA *is not constrained ordinally efficient.*

Proof. Let $N = \{i, j, k, z\}$ and $O = \{a, b, c, d\}$. The agents' preference profile is as follows:

$$R_i: \; a, b, c, d; \; R_j: \; d, b, a, c; \; R_k: \; b, a, c, d; \; R_z: \; d, c, a, b.$$[31]

The priority order profile of objects Δ is such that $supp(\Delta)$ consists of the following four deterministic profiles:

[29] Recall our initial supposition that $|N| = |O|$.

[30] Similarly, in the deterministic and coarse priority domain, Erdil and Ergin (2008) show that applying the DA after breaking ties is dominated by another stable matching. This result does not imply Proposition 3 here. The reason is that any Pareto superior stable matching to the DA in their setting is not necessarily claimwise stable in the corresponding random priority profile. On the other hand, as the DA does not satisfy the stability notions of Kesten and Ünver (2014), they do not consider its efficiency properties.

[31] The earlier an object comes, the more preferred it is. For instance, agent i prefers object a to object b, and so on.

$\succ$				$\succ'$				$\succ''$				$\succ'''$			
a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d
k	i	z	z	k	i	z	j	k	i	z	z	k	i	z	j
i	j	j	j	i	j	j	z	i	k	j	j	i	k	j	z
j	k	i	i	j	k	i	i	j	z	i	i	j	z	i	i
z	z	k	k	z	z	k	k	z	j	k	k	z	j	k	k

Let the probability of each of the above deterministic profiles be 1/4.[32] Then, the DA outcome is given as follows:

	a	b	c	d
i	3/4	1/4	0	0
j	0	0	1/2	1/2
k	1/4	3/4	0	0
z	0	0	1/2	1/2

Now, consider the following matching σ:

	a	b	c	d
i	1	0	0	0
j	0	0	1/2	1/2
k	0	1	0	0
z	0	0	1/2	1/2

It is easy to verify that σ is claimwise stable and ordinally dominates $DA(R, \Delta)$. Therefore, the DA is not constrained ordinally efficient.

$\square$

Remark 4. The DA dominates any other stable mechanism in the conventional deterministic domain. Nevertheless, in the current model, it turns out not to be even constrained

[32]Even though we do not rule out correlated priorities, they are independent in this example.

ordinally efficient under a natural generalization of stability. What is the intuition behind this sharp difference? In the course of the DA, an agent is rejected from an object whenever any higher priority one applies to it, regardless of the latter's assignments at the other deterministic priorities in support of Δ. That is, the DA outcome at any deterministic priority order is independent of those at the other deterministic priority orders in support of Δ. Claimwise stability constraints, whereas, relax as agents obtain more of their preferred objects (because of the second term on the right-hand side of the claimwise stability constraint; see Definition 1). As explained above, the DA fails to internalize this relaxation, resulting in efficiency loss beyond what caused by claimwise stability.

3.2 The Constrained Probabilistic Serial Mechanism (CPS)

Given the efficiency shortcoming of the DA, we introduce a new mechanism built on the PS of Bogomolnaia and Moulin (2001). We first show that it is both claimwise stable and constrained ordinally efficient, unlike the DA. We then compare the DA with our new mechanism in terms of the strategic and fairness properties.

First, we describe the PS of Bogomolnaia and Moulin (2001). Over the unit time interval, agents continuously consume objects at the speed of one in decreasing order of their preferences. The consumed shares by time $t = 1$ are the agents' PS assignment. Bogomolnaia and Moulin (2001) show that in contrast to the well-known random priority mechanism, the PS is ordinally efficient and satisfies some other desirable properties. Since their study, the PS has received a great deal of attention in the literature.

The PS has been introduced for priority-free object allocation problems; hence, it is not claimwise stable.[33] Below, we modify the PS to accommodate claimwise stability constraints.

Similar to the PS, in the course of the CPS, agents continuously acquire shares of objects at the speed of one in decreasing order of their preferences. The only difference between the

[33]For instance, let $N = \{i, j\}$ and $O = \{a, b\}$ with $R_i = R_j : a, b$. Let the priorities be deterministic such that agent i has higher priority than agent j for both of the objects. Then, $PS_{i,a} = PS_{i,b} = PS_{j,a} = PS_{j,b} = 1/2$, which is not claimwise stable.

mechanisms is the rule governing when agents stop consuming objects. In the course of the CPS, an agent stops consuming an object whenever relevant claimwise stability constraints start binding or the object is totally exhausted (whichever occurs first). The algorithm moves to a new step whenever any agent stops consuming an object, and it terminates when all objects are totally exhausted.

Before giving the formal definition of the algorithm, we illustrate how it works in a simple example. To this end, let us revisit the example in the proof of Proposition 3 so that we can also see the difference between the DA and CPS outcomes.

Step 1. Each agent first attempts to consume his favorite object. Agents j and z's best object is object d, and as $Pr_{\Delta}(j \succ_d z) = 1/2$, both of the agents are allowed to consume at most 1/2 of object d. As the other agents apply to different objects, no constraint is imposed on them in this round. This step, hence, terminates at $t^1 = 1/2$. By the end, the agents have consumed 1/2 of their respective objects, and object d has been exhausted.

Step 2. Agents j and z now attempt to consume their second best alternatives, which are objects b and c, respectively. This means that agents k and j try to consume the same object in this step, whereas the others continue to consume different objects. We therefore need to check the claimwise stability constraints for agents k and j. Given that $Pr_{\Delta}(k \succ_b j) = 1/2$ and object b is the top choice of agent k, agent j can consume at most 1/2 of object b (note that as agent j has already consumed 1/2 of his more preferred object in the previous step, no claimwise stability constraint is imposed on agent k due to agent j). As 1/2 of object b has remained from the previous round, this step terminates at $t^2 = 3/4$. By the end, the agents have consumed an additional 1/4 of their respective objects, and object b has been exhausted.

Step 3. Agents k and j now attempt to consume their next best alternative, which is object a. Hence, agents i, j and k all desire to consume object a in this step. As $Pr_{\Delta}(i \succ_a j) = 1$ and object a is the best alternative of agent i, agent j is not allowed to acquire any positive amount of object a. On the other hand, $Pr_{\Delta}(k \succ_a i) = 1$; however, since agent k

has already consumed 3/4 of his more preferred objects, agent i can consume at most 3/4 of object a. Since he has already acquired 3/4 of object a, he cannot consume more of it. This step, hence, terminates at $t^3 = t^2 = 3/4$. By the end, agents i and j have been rejected from object a, and no one has consumed any positive amount.

Step 4. Agents i and j now attempt to consume their next best available object, which is object c (recall that object b has already been exhausted). Hence, agents i, j and z all desire to consume object c in this step. Since all of them have already consumed at least 1/2 of their more preferred objects, they all continue to consume object c until it is totally exhausted. Hence, this step terminates at $t^4 = 1$. By the end, the agents have consumed an additional 1/4 of their respective objects, and all the objects have been totally exhausted.

Below is the CPS outcome:

	a	b	c	d
i	3/4	0	1/4	0
j	0	1/4	1/4	1/2
k	1/4	3/4	0	0
z	0	0	1/2	1/2

Now, we give the formal definition of the algorithm. Each agent i is endowed with a unit eating speed function w_i, i.e., for any $t \in [0,1]$, $w_i(t) = 1$, where $w_i(t)$ is the speed at which agent i consumes an object at time t.

Given a $n \times n$ matrix σ, $A = (A_i)_{i \in N}$ where $A_i \subseteq O$, a priority order profile Δ, below we define Θ_i^a for each agent $i \in N$ and each object $a \in O$:

$$\Theta_i^a(\sigma, \Delta, A) = min\{Pr_\Delta(i \succ_a j) + \textstyle\sum_{c \in SU(R_j,a)} \sigma_{j,c} - \sigma_{i,a} : \forall j \in N \setminus \{i\} \text{ such that } aR_jb \;\; \forall\, b \in A_j\}.$$

$$(\Theta_i^a(\sigma, \Delta, A) = 1 \text{ if the above set is empty}).$$

Θ_i^a will keep track of how much more agent i can consume object a within each step of the algorithm without violating claimwise stability constraints.

Let $top(i, A_i)$ be the favorite object of agent i in A_i. We define $M(a, A, N) = \{i \in N : \ a = top(i, A_i)\}$. Let $A^0 = (A_i^0)_{i\in N}$ where $A_i^0 = O$ for each $i \in N$, $c^0 = 0$, and $\sigma^0 = [0]$ (matrix of zeros). Suppose that A^{s-1}, c^{s-1}, and σ^{s-1} are already defined. Then, for any $a \in \cup_{i\in N} A_i^{s-1}$:

$$y^s(a) = min\{y : \ \sum_{i\in M(a,A^{s-1},N)} \int_{c^{s-1}}^{y} w_i(t)\mathrm{d}t + \sum_{i\in N} \sigma_{i,a}^{s-1} = 1\}$$
$$(y^s(a) = +\infty, \text{ if } M(a, A^{s-1}, N) = \emptyset).$$

Define now,

$$y^s = min\{y^s(a) : \ \forall a \in \cup_{i\in N} A_i^{s-1}\}$$
$$\theta^s = min\{\Theta_i^{top(i,A_i^{s-1})}(\sigma^{s-1}, \Delta, A^{s-1}) : \ \forall \ i \in N\}$$
$$c^s = min\{y^s, \theta^s\}$$

$E = \{a \in \cup_{i\in N} A_i^{s-1} : \ c^s = y^s(a)\}$.

$$A_i^s = \begin{cases} A_i^{s-1} \setminus E \cup top(i, A_i^{s-1}) & \text{if } c^s = \Theta_i^{top(i,A_i^{s-1})}(\sigma^{s-1}, \Delta, A^{s-1}) \\ A_i^{s-1} \setminus E & \text{otherwise} \end{cases}$$

For each $i \in N$:

$$\sigma_{i,a}^s = \begin{cases} \sigma_{i,a}^{s-1} + \int_{c^{s-1}}^{c^s} w_i(t)\mathrm{d}t & \text{if } a = top(i, A_i^{s-1}) \\ \sigma_{i,a}^{s-1} & \text{otherwise} \end{cases}$$

As the algorithm moves to a next step when an agent stops consuming an object and everything (i.e., both agents and objects) is finite, the algorithm terminates in finite steps. The generated outcome σ^s at the termination step s is the final CPS outcome. In Theorem 1, we show that σ^s is indeed a matching, validating that the algorithm produces a matching.

Kesten and Ünver (2014) introduce a fractional deferred acceptance algorithm (FDA) that produces a strongly ex-ante stable matching that is ordinally dominant within the class of strongly ex-ante stable matchings. The FDA is mainly based on the DA, with two main differences: (i)

agents apply to objects for their remaining fractions, and (ii) agents may have the same priority at an object, and the FDA is defined in a way that such agents receive the same treatment. While the latter property of the FDA ensures equal treatment for equal priority students (with possibly different preferences), it entails efficiency losses. Their second mechanism, the fractional deferred acceptance algorithm and trading process ($FDAT$), recovers such efficiency losses at the expense of that property. The $FDAT$ is a two-step mechanism in which the FDA constitutes the first step and efficiency improvement cycles that preserve ex-ante stability are implemented in the second stage. They show that the $FDAT$ is ex-ante stable, is efficient within its class, and treats students with the same preferences and priorities equally.

Their mechanisms and the CPS are different. First of all, each accommodates different constraints (as already discussed, their stability notions and claimwise stability are independent of each other), making their outcomes different. In terms of how they work, both the FDA and the $FDAT$ are based on the DA, while the CPS is based on the PS. Therefore, while each step assignment is permanent in the CPS, it is tentative in both the FDA and the $FDAT$. Indeed, in the special domain in which every agent has the same priority at any object (which effectively means the priority-free setting as in the PS's setting), Kesten and Ünver (2014) show that both the FDA and the $FDAT$ produce different outcomes than the PS. The CPS, on the other hand, coincides with the PS in that domain (see Proposition 5).

Budish et al. (2013) introduce a generalized probabilistic serial mechanism that accommodates multi-unit allocations and various real-life constraints such as group-specific quotas in the school choice setting or curriculum constraints in course allocations. Their setting is priority-free, like that of Bogomolnaia and Moulin (2001), and their constraints basically specify the lowest and highest amounts of an object that can be assigned to a group of agents collectively. The generalized probabilistic serial mechanism allows agents to consume objects as long as such constraints permit. While our mechanism and theirs are constructed in methodologically similar ways, they are independent of each other because the constraint structures they accommodate are different.

In the following, we first show that the CPS coincides with the DA and the PS in their respective domains in which they admit good properties. We then give our first theorem in the whole domain.

Proposition 4. *$CPS(R, \succ) = DA(R, \succ)$ at every $(R, \succ) \in \mathcal{R}^{|N|} \times \zeta$.*

Proof. See the Appendix. $\square$

In words, the above result states that the CPS coincides with the DA in the conventional deterministic priorities domain.

The PS is defined for the priority-free object allocation problem, and it is shown that the PS is ordinally efficient, envy-free, and weakly strategy-proof. In the current framework, on the other hand, one can interpret any problem with Δ in which any strict priority is equally likely (implying that for any pair of agents i and j and object a, $Pr_{\Delta}(i \succ_a j) = 1/2$) as a priority-free problem. Therefore, it is desirable for the CPS to coincide with the PS in that subdomain, and this is indeed the case, as shown below.

Proposition 5. *For any R, $CPS(R, \Delta) = PS(R, \Delta)$ whenever $\Delta(\succ) = \Delta(\succ')$ for any $\succ, \succ' \in \zeta$.*

Proof. See the Appendix. $\square$

We now return to our full domain. With the following result, we show not only that the CPS is well-defined in the sense that it always produces a matching, but also that its outcome is always claimwise stable and efficient among the set of all claimwise stable matchings. In other words, the CPS is constrained ordinally efficient, in contrast to the DA.

Theorem 1. *The CPS is claimwise stable and constrained ordinally efficient.*

Proof. See the Appendix. $\square$

3.3 Strategic Properties

Definition 2. *A mechanism ψ is strategy-proof if at every problem instance (R, Δ) and for any $i \in N$ and $R'_i \in \mathcal{R}$, either $\psi_i(R, \Delta)$ ordinally dominates $\psi_i(R'_i, R_{-i}, \Delta)$ or $\psi_i(R, \Delta) = \psi_i(R'_i, R_{-i}, \Delta)$.*[34]

In words, under a strategy-proof mechanism, no agent can ever benefit by misreporting his preference, regardless of his cardinal utilities. While Bogomolnaia and Moulin (2001) show that the PS is not strategy-proof, it satisfies the weaker version of strategy-proofness discussed below.

[34] R_{-i} stands for agents' preferences excluding that of agent i.

Definition 3. *A mechanism ψ is weakly strategy-proof if at every problem instance (R, Δ) and for any $i \in N$ and $R'_i \in \mathcal{R}$, $\psi_i(R, \Delta)$ is not ordinally dominated by $\psi_i(R'_i, R_{-i}, \Delta)$.*

In contrast to strategy-proofness, the above weakening allows beneficial misreporting for some (but not all) cardinal utilities that represent agents' ordinal preferences.

From Dubins and Freedman (1981) and Roth (1982), we know that the DA is strategy-proof in the conventional deterministic setting. Given this result, it is straightforward to see that the DA continues to be strategy-proof in the current framework as well.

Proposition 6. *The DA is strategy-proof.*

In spite of its robust strategic properties, the main disadvantage of the DA is its lack of constrained efficiency. On the other hand, the result below reveals that claimwise stability and constrained ordinal efficiency are incompatible even with weak strategy-proofness. Hence, rather than constrained inefficiency being a problem of the DA, there is a general tension among these three properties. For the proof of the following result, we follow the same analytical approach as that used to obtain similar impossibility results in the literature. Namely, we provide a problem instance (R, Δ) at which every claimwise stable and constrained ordinally efficient rule is manipulable.[35]

Theorem 2. *Suppose $|N| \geq 4$. Then, there is no claimwise stable mechanism that is constrained ordinally efficient and weakly strategy-proof.*

Proof. See the Appendix. □

What is the intuition behind the above impossibility result? In order for a mechanism to achieve constrained ordinal efficiency, it must keep track of how much agents consume from their preferred objects (recall from Definition 1 that claimwise stability constraints at any object relax as agents acquire shares of their more preferred objects) and incorporate this into its dynamics. However, this necessary process depends on preferences giving room to agents to manipulate claimwise stability constraints in favor of themselves through misreporting their preferences.

As the CPS is claimwise stable and constrained ordinally efficient, Theorem 2 immediately implies that the CPS is not weakly strategy-proof.

[35]In Footnote 44, we mention a general form of Δ for which the impossibility result holds.

Corollary 2. *Suppose $|N| \geq 4$. Then, the CPS is not weakly strategy-proof.*

Truncation strategies (see, e.g., Roth and Vate (1991)), where agents report certain objects as unacceptable while keeping the same relative ordering of the other objects, are well-studied in the literature (let us assume the existence of the null object for the truncation strategies part). They are relatively easy strategies in the sense that agents need less information to manipulate through truncation (for a formal argument regarding this point, see Roth and Rothblum (2011)). Indeed, it is documented that agents have truncated their preferences in some real-life matching problems (see, e.g., Mongell and Roth (1991)). Fortunately, as shown below, the CPS is non-manipulable through truncation.[36]

Formally, for any student i, R_i' is a truncation of R_i if for some object $a \in O$, $bR_i'c \Leftrightarrow bR_ic$ for any $b, c \in U(R_i, a)$ and $\emptyset R_i' d$ for every $d \in O \setminus U(R_i, a)$.

Corollary 3. *Given any problem (R, Δ), agent i, and a truncation R_i' of R_i, either $CPS_i(R, \Delta)$ ordinally dominates $CPS_i(R_i', R_{-i}, \Delta)$ or $CPS_i(R, \Delta) = CPS_i(R_i', R_{-i}, \Delta)$.*

The proof is straightforward from the definition of the CPS. It is easy to see that if someone truncates his preferences, then he receives the same amount of the objects that are still acceptable under the truncation. On the other hand, he does not obtain any positive amount of the objects that are actually acceptable but reported as unacceptable under the truncation. Taken together, these show the result.

Remark 5. Recent literature shows that the scope of manipulations may diminish as the market becomes large.[37] In particular, Kojima and Manea (2010) show that the PS becomes strategy-proof in large problems. As the CPS is a variant of the PS, there is a sense that it may become (weakly) strategy-proof in large problems as well. While this paper does not pursue a large market analysis, this might be a fruitful direction for future research.

[36] The extension of the CPS to the presence of the null object case is straightforward.

[37] In the one-to-one matching context, Immorlica and Mahdian (2005) show that no agent has an incentive to misreport in expected terms whenever any other agent is truthful in large markets. This result is then extended to the many-to-one matching problem by Kojima and Pathak (2009). Afacan (2014) introduces a new kind of manipulation in the school choice context and shows that the DA becomes immune to that kind of manipulation as the problem is getting large.

3.4 Further Fairness Properties

We have already shown that both the CPS and the DA are claimwise stable, which is a fairness property. In this section, we investigate further fairness properties of the mechanisms.

Definition 4.

(i) A mechanism ψ is envy-free if at every problem instance (R, Δ) and for every pair of agents i, j, either $\psi_i(R, \Delta)$ ordinally dominates $\psi_j(R, \Delta)$ with respect to R_i or $\psi_i(R, \Delta) = \psi_j(R, \Delta)$.

(ii) A mechanism ψ satisfies equal-treatment of equals if at every problem instance (R, Δ) and for every pair of agents i, j such that $R_i = R_j$, $\psi_i(R, \Delta) = \psi_j(R, \Delta)$.

First, it is easy to see that both the CPS and the DA do not satisfy either of the above properties. This negative result is indeed very expected as agents are prioritized in our setting. Therefore, in the following, we first adopt the properties to our setting in a natural way.

Definition 5.

(I) A mechanism ψ satisfies limited envy-freeness if for any pair of agents i, j, $\psi_i(R, \Delta)$ is not ordinally dominated by $\psi_j(R, \Delta)$ with respect to R_i at every problem (R, Δ) such that for each $a \in O$ and $k \in N$, (i) $Pr_\Delta(i \succ_a j) \geq 1/2$ and (ii) $Pr_\Delta(i \succ_a k) \geq Pr_\Delta(j \succ_a k)$.[38]

(II) A mechanism ψ satisfies limited equal-treatment of equals if for any pair of agents i, j, $\psi_i(R, \Delta) = \psi_j(R, \Delta)$ at every problem (R, Δ) such that for each $a \in O$ and $k \in N$, (i) $R_i = R_j$, (ii) $Pr_\Delta(i \succ_a j) = 1/2$, and (iii) $Pr_\Delta(i \succ_a k) = Pr_\Delta(j \succ_a k)$.

Proposition 7. *Suppose $|N| \geq 3$. Then,*

(i) neither the CPS nor the DA satisfies limited envy-freeness;

(ii) while the CPS satisfies limited equal-treatment of equals, the DA does not.

Proof.

(i) We first show that the CPS does not satisfy limited envy-freeness. Let $N = \{i, j, k\}$ and $O = \{a, b, c\}$. The preference profile of agents is as follow:

[38]This condition is weaker than the envy-freeness in two ways. It not only depends on the priorities, but also require no agent's assignment to be ordinally dominated by someone else's assignment (as opposed to envy-freeness requiring each agent's assignment to ordinally dominate (or be the same as) every other's assignment).

$$R_i: \ a, b, c; \ R_j: \ b, a, c; \ R_k: \ a, c, b.$$

Let $Pr_\Delta(s \succ_c s') = 1/2$ for any $s, s' \in N$, and let the priorities of the other objects be as follows:

$\succ$		$\succ'$		$\succ''$	
a	b	a	b	a	b
i	i	k	i	k	j
j	j	i	j	j	i
k	k	j	k	i	k

Let $\Delta(\succ) = 1/8$, $\Delta(\succ') = 3/8$, and $\Delta(\succ'') = 1/2$. As we can see, $Pr_\Delta(i \succ_d j) = 1/2$, and $Pr_\Delta(i \succ_d k) = Pr_\Delta(j \succ_d k)$ for all $d \in O$. Therefore, the limited envy-freeness conditions are satisfied. Then, $CPS(R, \Delta)$ is given below:

	a	b	c
i	1/8	7/16	7/16
j	1/8	9/16	5/16
k	3/4	0	1/4

It is easy to verify that $CPS_j(R, \Delta)$ ordinally dominates $CPS_i(R, \Delta)$ with respect to R_i. Therefore, the CPS does not satisfy limited envy-freeness.

Next, we provide a problem instance at which the DA does not satisfy both limited envy-freeness and limited equal-treatment of equals. Let $N = \{i, j, k\}$ and $O = \{a, b, c\}$. Assume that $R_i = R_j : a, b, c$; and $R_k : b, a, c$. The priority profiles in $supp(\Delta)$ are as follows:

$\succ$			$\succ'$		
a	b	c	a	b	c
k	k	k	j	j	j
i	i	i	i	i	i
j	j	j	k	k	k

Let $\Delta(\succ) = \Delta(\succ') = 1/2$. Note that agents i and j satisfy the assumptions of limited envy-freeness and limited equal-treatment of equals. Then, $DA_{i,a}(R, \Delta) = DA_{i,b}(R, \Delta) = 1/2$ and

$DA_{j,a} = DA_{j,c} = 1/2$. It is easy to verify that DA_i ordinally dominates DA_j; hence the DA satisfies neither limited envy-freeness nor limited equal-treatment of equals.

(ii) We have already shown that the DA does not satisfy limited equal-treatment of equals; hence, we only need to prove that the CPS satisfies it. Given any problem (R, Δ), let us assume that there are agents i, j such that for each $a \in O$ and $k \in N$, (i) $R_i = R_j$, (ii) $Pr_\Delta(i \succ_a j) = 1/2$, and (iii) $Pr_\Delta(i \succ_a k) = Pr_\Delta(j \succ_a k)$.

Let us enumerate the objects in $O = \{a_1, a_2, .., a_k\}$ such that $a_1 P_i a_2, ..$ and so on. Then, by the construction of the CPS, both the agents i and j first attempt to consume object a_1. Given that the agents consume the objects at the same unit speed in the course of the CPS, $CPS(R, \Delta)_{i,a_1} = CPS(R, \Delta)_{j,a_1}$. This means that they attempt to consume their second favorite object at the same time. Then, by the same logic, we have $CPS(R, \Delta)_{i,a_2} = CPS(R, \Delta)_{j,a_2}$. The same argument for all the other remaining objects shows that $CPS_i(R, \Delta) = CPS_j(R, \Delta)$, which finishes the proof.

□

3.5 A Characterization of the CPS

In this section, we provide an axiomatic characterization of the CPS in a more general setting than what we started with. Here, we assume the existence of the null object. The extension of the CPS to the presence of the null object case is straightforward.

As we mentioned before, there has been a recent surge in the characterization of the PS of Bogomolnaia and Moulin (2001): Bogomolnaia and Heo (2012) and Hashimoto et al. (2014) provide different characterizations of the PS.[39] In particular, Bogomolnaia and Heo (2012) introduce the consumption process representation of random assignments, which has proven very useful in the characterization of the PS.

In obtaining our axiomatization, we invoke that consumption process representation as well. Even though we described it earlier in the paper, for the sake of completeness, we outline it again here. Each matching can be considered as the outcome of the consumption process through which agents continuously acquire shares of objects in decreasing order of their preferences at the speed

[39] The latter merges two previous working papers: Hashimoto and Hirata (2011) and Kesten et al. (2011).

of one. In the course of this process, agents' switching times between objects generate all different matchings. For instance, in the PS, agents continue to consume their preferred objects until they are exhausted. On the other hand, in the CPS, they do so until either the corresponding claimwise stability constraints become binding or the objects are totally exhausted (whichever occurs first). In the rest of this subsection, we associate each matching with its corresponding consumption process, and we use the same notation for both.

Before moving to the axioms, we define the steps of the consumption process and put them in order. In the course of the consumption process, agents first attempt to consume their best objects; this constitutes the initial step. Then, whenever an agent stops consuming his current object and attempts to consume his best remaining object, the process moves to the next step, and so forth. One caveat in the ordering of the steps is that multiple steps might occur at the same time. Indeed, an agent might apply to different objects successively in decreasing order of his preference at the same time instant.

In order to overcome the above problem, we add another time dimension to the consumption process. It ticks away with the steps as opposed to the usual time, which ticks away as agents acquire objects' shares. Formally, for a given mechanism ψ and a problem (R, Δ), let S be the set of steps occurring in the course of the consumption process $\psi(R, \Delta)$. Below, we suppress the dependency of S and ψ on the problem instance. Let us consider the set of discrete artificial time instances $T = \{1, 2, ...n\}$, where $|T|$ is equal to the number of steps. We then assign an artificial time index to each step. Formally, let $\iota : S \rightarrow T$ be a bijective function, where $\iota(s)$ is the artificial time index assigned to step s. As agents consume objects in decreasing order of their preferences, we assume that $\iota(s) < \iota(s')$ for any steps s, s' in which the same agent attempts to consume different objects and prefers the one corresponding to step s. Apart from that, there is no other restriction on ι. Moreover, we write $s(t)$ for the usual time instance at which step s starts. We then define an order $\prec$ on S as follows:

$$\text{For any given } s, s' \in S,\ s \prec s' \text{ if either } s(t) < s'(t) \text{ or } s(t) = s'(t) \ \&\ \iota(s) < \iota(s').$$

In words, if $s \prec s'$, then we say that step s occurs before step s' in the course of ψ. By introducing ι, we order the steps occurring at the same time and let them happen successively according to the order $\prec$. Note that the consumption process outcome is independent of the ordering of the steps occurring at the same time. It is easy to see that $\prec$ is a strict, complete, and transitive binary relation on S.

For a given matching σ, agent i, and object a, Kesten et al. (2011) and, subsequently, Hashimoto et al. (2014) define $F(R_i, a, \sigma_i) = \sum_{c \in U(R_i,a)} \sigma_{i,c}$ They call it "agent i's surplus at object a under σ_i." Then, they introduce the following axiom.

Ordinal Fairness (Kesten et al. (2011) and Hashimoto et al. (2014)): A mechanism ψ is ordinally fair if at every problem instance R,[40] there are no pair of agents i, j and object a such that $F(R_i, a, \psi_i) < F(R_j, a, \psi_j)$ and $\psi(R, \Delta)_{j,a} > 0$.

A mechanism ψ is *non-wasteful* if, for any problem (R, Δ) and any agent i, aP_ib for some b and $\psi(R, \Delta)_{i,b} > 0$, then $\sum_{j \in N} \psi(R, \Delta)_{j,a} = 1$.

Kesten et al. (2011) and Hashimoto et al. (2014) show that the PS is the only mechanism that is non-wasteful and ordinally fair. In our setting, on the other hand, it is easy to see that claimwise stability and ordinal fairness are incompatible. This is very expected because agents are not equal in the current environment (they are prioritized), and ordinal fairness is a kind of equality property. In the following, we adopt ordinal fairness in our priority-based setting.

Definition 6. *Mechanism ψ is binding if at any problem (R, Δ) and for any pair of agents i and j and object a, $F(R_i, a, \psi_i) < F(R_j, a, \psi_j)$ and $\psi(R, \Delta)_{j,a} > 0$, then there exists an agent k such that the followings hold:*

(i) $\psi(R, \Delta)_{i,a} = Pr_\Delta(i \succ_a k) + \sum_{c \in SU(R_k,a)} \psi(R, \Delta)_{k,c}$, and

(ii) The steps in the consumption process $\psi(R, \Delta)$ can be ordered in a way that if s and s' are the steps in which agent k applies to object a and agent i stops consuming it, respectively, then $s \prec s'$.

For the justification of the above property, we first observe that at any matching (or consumption process) σ, if $F(R_i, a, \sigma_i) < F(R_j, a, \sigma_j)$ and $\sigma_{j,a} > 0$, they imply that agent i stops consuming object a before it is totally exhausted. Under a binding mechanism, it happens only if some agent k (agents k and j might be the same) applies to object a in a step before agent i stops consuming

[40] No priority in their setting.

it, and the claimwise stability constraint imposed on the latter by the former is binding. That is, bindingness requires any agent to continue consuming his preferred object (unless it is exhausted) until someone else applies to it and prevents him from consuming it through claimwise stability constraints. Therefore, we can say that agents are doing their best to consume their preferred objects under binding mechanisms.

In order to understand the role of the ordering of the steps in the above axiom, consider a problem consisting of $N = \{i, j\}$ and $O = \{a, b\}$. The preference profile of agents is as follows: $R_i : a, b$; and $R_j : b, a$. The priority orders of objects are deterministic such that $j \succ_a i$ and $i \succ_b j$. Then, consider the matching σ under which $\sigma_{i,b} = 1$ and $\sigma_{j,a} = 1$. That is, agent i is assigned to object b and agent j is assigned to object a. It is easy to see that σ is claimwise stable. However, it is not binding. In order to see this, first observe $F(R_i, a, \sigma_i) < F(R_j, a, \sigma_j)$ and $\sigma_{j,a} > 0$; and similarly, $F(R_j, b, \sigma_j) < F(R_i, b, \sigma_i)$ and $\sigma_{i,b} > 0$. Then, in order for the matching σ to be binding, the two conditions in Definition 6 have to hold. It is easy to verify that the claimwise stability constraints bind at the matching σ. Hence, we need to be able to order the steps in the consumption process σ in the aforementioned way. Let s_1 and s_2 be the steps in which agents i and j, respectively, stop consuming their best objects and apply to their second best alternatives. It has to be that $s_1 \prec s_2$, yet at the same time, $s_2 \prec s_1$. This contradicts the definition of $\prec$; therefore, σ is not binding. In words, this shows that agents are not doing their best to consume their preferred objects; that is, either (or both) of them voluntarily stops consuming his best object in the course of the consumption process σ.

Before the characterization result, we give the following lemma, which will be used in the proof of Theorem 3, and which is of interest on its own.

Lemma 1. *A constrained ordinally efficient matching is non-wasteful.*

Proof. See the Appendix. □

Theorem 3. *A mechanism ψ is non-wasteful, claimwise stable, and binding if and only if it is the CPS.*

Proof. See the Appendix. □

Remark 6. If $|N| \geq |O|$ (this nests the absence of the null object case), we do not need non-wastefulness, and the characterization is given by claimwise stability and bindingness.

In the following examples, we show the independence of the axioms.

Example 1: From Hashimoto et al. (2014), we know that the PS is non-wasteful, ordinally fair, hence binding. However, it is not claimwise stable whenever $|N| \geq 2$.

Example 2: The mechanism leaving every agent unassigned is claimwise stable and binding, yet wasteful.

Example 3: If $|N| \geq 2$ and $|O| \geq 2$, then a non-wasteful and claimwise stable mechanism is not necessarily binding. To see this, consider the example given immediately prior to Theorem 3 above. The matching instance given there is non-wasteful and claimwise stable, yet it is not binding.

Remark 7. Erdil (2014) shows that the DA is wasteful. Moreover, it is not binding as well. This can be seen via the DA outcome given in Proposition 3. There, $F(R_j, b, DA_j) < F(R_i, b, DA_i)$, where $DA_{i,b} > 0$. It is easy to verify that the first condition in the axiom does not hold.

4 Extensions

4.1 Different Numbers of Objects and Agents and the Presence of the Null Object

We presented the results for the case of $|N| = |O|$. However, if $|N| > |O|$, then nothing will change in our analysis above; all the results carry over. On the other hand, if $|N| < |O|$ (note that this case nests the presence of the null object), then wastefulness will be an issue. That is, there might be claimwise stable matchings that are wasteful. For instance, the matching at which every agent is unassigned would be claimwise stable. Another direct example is the DA, which is claimwise stable yet wasteful (Erdil (2014)). On the other hand, from the characterization section, we know that the CPS is non-wasteful in this more general case (we show it under the presence of the null object, yet it easily holds in the case of $|N| < |O|$ without the null object). As it can be easily argued that wasteful objects might be interpreted as justified claims against the social planner, we also require non-wastefulness from claimwise stable matchings. In this case, the DA is not even claimwise stable, while all the results regarding the CPS carry over.

4.2 Multi-Quota Case

In our analysis, we assume that there is only one copy (unit quota) from each object (except the null object). However, we pointed out that this is just a simplification assumption; our results hold for the multi-quota case as well.

In the multi-quota case, the definition of claimwise stability is problematic. For example, consider a problem consisting of $N = \{i, j\}$ and $O = \{a\}$, where $q_a = 2$. Assume that $R_i = R_j : a, \emptyset$; and $Pr_\Delta(i \succ_a j) = 1$. Then, under any claimwise stable matching σ: $\sigma_{j,a} = 0$ even if $\sigma_{i,a} = 1$. This obviously does not make sense, as agent i is already assigned to his favorite object with certainty; therefore, giving the other copy to agent j would not be unfair to agent i in any sense.

In order to overcome the above caveat, we consider each copy of an object to be a different object endowed with the originally given priority order. For instance, if we have two copies of object a, then we consider the second copy to be a different object, say a', and its priority order is the same as that of object a. The next question in this construction is how we define the preferences of agents over the artificially created larger set of objects. Given two objects a and b that are not the copies of the same object, if R'_i denotes the artificially created preferences of agent i, then aR'_ib if and only if aR_ib. We can define R'_i over the copies of the same objects in any way. Now, through this artificial construction, we can transform every multi-unit quota problem to a unit quota problem and solve the latter instead of the former. Below, we see how our construction solves the problem in the above instance.

Let us now consider the problem as if there are two objects, a and a', with $q_a = q_{a'} = 1$. For the preference profile of agents, construct the new preferences as follows: $R'_i = R'_j : a, a', \emptyset$. Moreover, $Pr_\Delta(i \succ_a j) = Pr_\Delta(i \succ_{a'} j) = 1$. Then, the random assignment σ where $\sigma_{i,a} = 1$ and $\sigma_{j,a'} = 1$ is claimwise stable.[41]

Given that we transform multi-quota problems to unit-quota ones, all the results also apply to the multi-quota setting as well.

[41] It is the efficient claimwise stable matching.

5 Conclusion and Future Research

This paper studies the object allocation problem with random priorities for the first time in the literature. We first introduce a fairness notion called claimwise stability. While the well-known DA turns out to be claimwise stable, it is dominated by another claimwise stable rule. Given this important shortcoming of the DA, we adopt the PS of Bogomolnaia and Moulin (2001) to the current setting and introduce the CPS. It is both claimwise stable and constrained ordinally efficient. Then, we compare the DA and the CPS in terms of the strategic and fairness properties. Lastly, we provide a characterization of the CPS. The following table summarizes our comparison between the CPS and the DA.

	DA	CPS
Claimwise stability	✓	✓
Constrained Ordinal Efficiency	✗	✓
Non-wastefulness	✗	✓
Strategy-proofness	✓	✗
Weak Strategy-proofness	✓	✗
Limited Envy-freeness	✗	✗
Limited Equal-treatment of Equals	✗	✓
Bindingness	✗	✓

While the CPS achieves constrained ordinal efficiency and exhibits better fairness properties than the DA, the converse is true with respect to strategic issues. The CPS is built on the PS, and there are other variants of the PS performing well in different settings. This paper strengthens the idea that adopting the PS to other environments might be a fruitful direction for future research.

Appendix

Proof of Proposition 1. Let σ be a claimwise stable matching at problem $(R, \succ)$. If σ is deterministic, then it is easy to see that σ is stable. Let us assume that it is random. Consider an agent i and object a such that $\sigma_{i,a} > 0$. Then, as σ is claimwise stable, any agent j who has higher priority than agent i at object a receives at least $\sigma_{i,a}$ share of his strictly preferred objects. This implies that matching σ can be written as a lottery over deterministic matchings under which whenever agent i is matched with object a, then agent j is matched with one of his more preferred objects. As this is true for all of the agents and the objects, we can write σ as a lottery over stable matchings, showing that it is ex-post stable.

□

Proof of Proposition 4. In the course of the CPS, agents first apply to their respective favorite objects. All of the agents except the highest priority ones are rejected from their respective objects. Note that in the CPS (by its definition), no agent starts consuming an object until everyone is tentatively assigned. Next, the rejected agents in the previous round apply to their second best objects. Similar to above, all of the agents except the highest priority ones among the current applicants and the tentatively assigned ones are rejected from their respective objects, and so on and so forth. Whenever everyone is tentatively assigned to some object, they consume their assignments (note that as each agent is tentatively assigned to a different object, the final assignments are deterministic). This wordy explanation of the CPS shows that $CPS(R, \succ) = DA(R, \succ)$ for every problem $(R, \succ) \in \mathcal{R}^{|N|} \times \zeta$.

□

Proof of Proposition 5. If $\Delta(\succ) = \Delta(\succ')$ for any $\succ, \succ' \in \zeta$ (that is Δ is the uniform distribution over ζ), then it means that for any pair of students i, j and object a, $Pr_\Delta(i \succ_a j) = 1/2$. In words, any agent has the same priority; hence, everyone has the same claim on any object. Therefore, in the course of the CPS, whenever more than one agent desires to consume the same object, as claimwise stability allows each of them to have at least $1/2$ of it and they all consume the object with the same speed of one, they all continue to eat it until the object is totally exhausted. Moreover, no claimwise stability constraint is imposed on an agent while consuming an object whenever all the others are consuming their more preferred objects. Taken together, these show that in the CPS, no agent stops consuming an object before it is totally exhausted, showing that the CPS and the PS coincide with each other for any R whenever $\Delta(\succ) = \Delta(\succ')$ for any $\succ, \succ' \in \zeta$. □

Proof of Theorem 1. We first need to show that for any problem (R, Δ), $CPS(R, \Delta)$ is indeed a matching, i.e., $\sum_{i \in N} CPS(R, \Delta)_{i,a} = 1$ and $\sum_{a \in O} CPS(R, \Delta)_{i,a} = 1$ for all $a \in O$ and $i \in N$. For ease of notation, let $\sigma = CPS(R, \Delta)$. By the definition of the CPS, agent i continues to consume until $\sum_{a \in O} \sigma_{i,a} = 1$. On the other hand, as $|N| = |O|$ and $q_a = 1$ for every $a \in O$, it also implies that $\sum_{i \in N} \sigma_{i,a} = 1$ for every $a \in O$.

Now, we claim that σ is claimwise stable. Assume for a contradiction that there exist agents i, j and an object a such that $\sigma_{i,a} > Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j, a)} \sigma_{j,c}$. Let s_0 and s_1 be the steps in which agent i starts consuming object a and stops consuming it, respectively. Then, by our supposition, $Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j, a)} \sigma^{s_1}_{j,c} - \sigma^{s_1}_{i,a} < 0$[42] (this is due to our supposition and the fact that $\sum_{c \in SU(R_j, a)} \sigma^{s_1}_{j,c} \leq \sum_{c \in SU(R_j, a)} \sigma_{j,c}$). Then, the construction of the CPS implies that there exists a step $\tilde{s} < s_1$ such that $Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j, a)} \sigma^{\tilde{s}}_{j,c} - \sigma^{\tilde{s}}_{i,a} = 0$. By the definition of Θ^a_i (see the formal description of the CPS), $Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j, a)} \sigma^{\tilde{s}}_{j,c} - \sigma^{\tilde{s}}_{i,a} = 0$ implies that agent i cannot consume object a after step $\tilde{s}$ in the course of the CPS, which contradicts our supposition. Hence, the CPS is claimwise stable.

Next, we claim that the CPS is constrained ordinally efficient. Assume for a contradiction that there exists a claimwise stable matching ϕ ordinally dominating σ (recall that $\sigma = CPS(R, \Delta)$). We first introduce some notations: Given a matching ψ, we define $F(R_i, a, \psi_i) = \sum_{c \in U(R_i, a)} \psi_{i,c}$. Let $\pi_1 = min\{F(R_i, a, \phi_i) : \text{for all } (i, a) \in N \times O\}$, $\pi_k = min\{F(R_i, a, \phi_i) : \; F(R_i, a, \phi_i) > \pi_{k-1} \text{ for all } (i, a) \in N \times O\}$. We also write $\Pi = \{F(R_i, a, \phi_i) : \text{for all } (i, a) \in N \times O\}$.

In what follows, we will show that $F(R_i, a, \phi_i) = F(R_i, a, \sigma_i)$ for all agent i and object a, which will in turn imply that $\phi = \sigma$. This will constitute a contradiction to our starting supposition that σ is ordinally dominated by ϕ, which will finish the proof. We prove it through induction. Let $k = 1$ and consider an agent-object pair (i, a) such that $F(R_i, a, \psi_i) = \pi_1$. We have two cases to consider.

Case 1 If $F(R_i, a, \phi_i) = 0$, then as ϕ ordinally dominates σ, we have $F(R_i, c, \sigma_i) = 0$ for all $c \in U(R_i, a)$. In particular, we have $F(R_i, a, \sigma_i) = 0$.

Case 2 Let us now consider that $F(R_i, a, \phi_i) > 0$ and object a is the top choice of agent i. If we

[42] σ^{s_1} is the assignment generated in step s_1 in the course of the CPS. Its formal definition is given in the formal description of the CPS.

write $a(j)$ for the favorite object of agent $j \in N$, then $F(R_j, a(j), \phi_j) \geq \pi_1$ (by the definition of π_1. Note that by our supposition, $a(i) = a$). On the other hand, by the definition of the CPS, all the agents first attempt to consume their favorite objects, and they continue to consume until either claimwise stability constraints start binding or the objects become totally exhausted (whichever occurs first). This, along with the claimwise stability of ϕ, and $F(R_j, a(j), \phi_j) \geq \pi_1$ for any j imply that no agent $i \in N$ stops consuming his favorite object before time $t = \pi_1$ in the course of the CPS. Hence, $F(R_j, a(j), \sigma_j) \geq \pi_1$ for any $j \in N$, in particular, $F(R_i, a, \sigma_i) \geq \pi_1$. This finding and our supposition that $F(R_i, a, \sigma_i) \leq F(R_i, a, \phi_i) = \pi_1$ show that $F(R_i, a, \sigma_i) = \pi_1$.

Now, let us assume that a is not the favorite object of agent i. Let us say that it is his second best object (the other cases follow from the same argument). Then, from the above analysis, we know that $\sigma_{i,a(i)} = \phi_{i,a(i)}$, where $a(i)$ is the favorite object of agent i. Therefore, $F(R_i, a, \sigma_i) \geq \pi_1$. On the other hand, by our supposition, $F(R_i, a, \phi_i) = \pi_1 \geq F(R_i, a, \sigma_i)$; hence, $F(R_i, a, \sigma_i) = F(R_i, a, \phi_i) = \pi_1$.

Let us assume that $F(R_i, a, \phi_i) = F(R_i, a, \sigma_i)$ for any agent-object pair (i, a) such that $F(R_i, a, \phi_i) \leq \pi_{k-1}$.

Let us consider an agent-object pair (i, a) such that $F(R_i, a, \phi_i) = \pi_k$. We want to show that $F(R_i, a, \sigma_i) = F(R_i, a, \phi_i)$. First, assume that $\phi_{i,a} > 0$. This implies that $F(R_i, b, \phi_i) \leq \pi_{k-1}$ where b is the object just preferred to object a by agent i. Let us write $F(R_i, b, \phi_i) = t$. By the definition of the CPS, agent i attempts to consume object a at time t (note that by the induction hypothesis, $F(R_i, b, \sigma_i) = t$). In what follows, we first show that agent i does not stop consuming object a before time $t' = \pi_{k-1}$ in the course of the CPS (note that by construction, $t \leq \pi_{k-1}$. If $t = \pi_{k-1}$, then no need to show this part).

Let $N(a) = \{j \in N : \ \phi_{j,a} > 0 \ \& \ F(R_j, a, \phi_j) \leq t'\}$. As $\sum_{j \in N} \phi_{j,a} = 1$ and $F(R_i, a, \phi_i) = \pi_k$ with $\phi_{i,a} > 0$ (so that $i \notin N(a)$), we have $\sum_{j \in N(a)} \phi_{j,a} < 1$. This fact, along with the induction hypothesis, implies that object a is not exhausted before time t' in the course of the CPS.

Therefore, the only reason that might stop agent i from consuming object a before time t' in the CPS is claimwise stability. Assume that he stops consuming object a before t', i.e., the claimwise stability constraints imposed on him for object a bind before t'. This implies that there exists an agent k such that $\sigma_{i,a} = Pr_{\Delta}(i \succ_a k) + \sum_{c \in SU(R_k, a)} \sigma_{k,c}$. By our supposition, $\sigma_{i,a} < t' - t$, where

$t' = \pi_{k-1}$. This implies that $\sum_{c \in SU(R_k,a)} \sigma_{k,c} < \pi_{k-1}$. Then, we have two cases to consider.

Case (i). If $\sum_{c \in SU(R_k,a)} \phi_{k,c} \leq \pi_{k-1}$, then by the induction hypothesis, we have $\sum_{c \in SU(R_k,a)} \sigma_{k,c} = \sum_{c \in SU(R_k,a)} \phi_{k,c}$. As $\phi_{i,a} > \sigma_{i,a}$, this case contradicts the claimwise stability of ϕ.

Case (ii). Let us consider the case where $\sum_{c \in SU(R_k,a)} \phi_{k,c} > \pi_{k-1}$. Recall that $F(R_i, b, \phi_i) = F(R_i, b, \sigma_i) = t < \pi_k$, where b is the object just preferred to object a by agent i. Let $\pi_{k'} \in \Pi$ be such that it comes just after t. That is, $t < \pi_{k'}$ and $\pi_{k'} \leq \pi_h$ for all $\pi_h \in \Pi$ such that $\pi_h > t$.[43] We first show that in the course of the CPS, agent i does not stop consuming object a before time $\tilde{t} = \pi_{k'}$. The proof is very similar to the proof of the base step of the induction. In the course of the CPS, any agent j who stops consuming an object at time t applies to his next best available object, say z. If $\phi_{j,z} = 0$, then $F(R_j, z, \sigma_j) = F(R_j, z, \phi_j) = t$ (by the induction hypothesis). Hence, for such an agent j, let us assume that $\phi_{j,z} > 0$. On the other hand, for all of the other agents, i.e, the ones who do not stop consuming their respective objects at time t continue to consume their respective objects. Now, in order to make analogy with the base step, we can consider the objects agents are consuming at time t as their best objects. Then, given that ϕ is claimwise stable and $\pi_{k'}$ comes just after t (by construction), no agent stops consuming his best object before $\tilde{t} = \pi_{k'}$ in the course of the CPS. In particular, agent i does not stop consuming object a before $\tilde{t} = \pi_{k'}$.

Now, let $\pi_{k''} \in \Pi$ come just after $\pi_{k'}$ (if $\pi_{k'} = \pi_{k-1}$, then no need to show this part). That is, $\pi_{k'} < \pi_{k''}$ and $\pi_{k''} \leq \pi_h$ for all $\pi_h \in \Pi$ such that $\pi_{k'} < \pi_h$. Let $t = \pi_{k'}$ and $\tilde{t} = \pi_{k''}$. Then, by the same arguments as above, we can easily show that agent i continues to consume object a between $\pi_{k'}$ and $\pi_{k''}$ in the course of the CPS. If we continue in the same manner till time $t' = \pi_{k-1}$, we eventually prove that agent i does not stop consuming object a before $t' = \pi_{k-1}$ in the course of the CPS.

We now claim that agent i continues to consume object a between times $t' = \pi_{k-1}$ and $t'' = \pi_k$ as well. By construction, π_k is the element of Π coming just after π_{k-1}. This part, therefore, directly follows from the same arguments as above again . Hence, $F(R_i, a, \sigma_i) \geq \pi_k$. Given that $\pi_k = F(R_i, a, \phi_i) \geq F(R_i, a, \sigma_i)$, we have $F(R_i, a, \sigma_i) = \pi_k$.

In the above analysis, we assume that $\phi_{i,a} > 0$. Let us now consider $\phi_{i,a} = 0$. As $F(R_i, a, \phi_i) = \pi_k$ and $\phi_{i,a} = 0$, there exists an object b such that bP_ia, $F(R_i, b, \phi_i) = \pi_k$, and $\phi_{i,b} > 0$. From above,

[43] Note that it might be that $\pi_{k'} = \pi_{k-1}$.

we have $F(R_i, b, \sigma_i) = F(R_i, b, \phi_i) = \pi_k$. Then, by our supposition, $\pi_k = F(R_i, a, \phi_i) \geq F(R_i, a, \sigma_i)$. This, along with $F(R_i, b, \sigma_i) = \pi_k \leq F(R_i, a, \sigma_i)$, implies that $F(R_i, a, \sigma_i) = \pi_k$, which finishes the proof.

□

Proof of Theorem 2. Consider a problem instance consisting of $N = \{i, j, k, z\}$ and $O = \{a, b, c, d\}$. The preference profile of agents is as follows:

$$R_i: \; b, a, c, d; \; R_j: \; b, a, d, c; \; R_k: \; b, d, a, c; \; R_z: c, b, d, a.$$

Let the random priority profile $\Delta = (\Delta_a)_{a \in O}$ be such that (i) the priorities of objects a, c, and d are deterministic such that

$$\succ_a: \; j, i, ..; \succ_c: \; i, z, ..; \succ_d: \; k, j, ...$$

Object b's priority order is random such that the support of Δ_b consists of the following deterministic priorities $\succ_b$ and $\succ'_b$ with $\Delta(\succ_b) = 1/6$ and $\Delta(\succ'_b) = 5/6$:[44]

$\succ_b$	$\succ'_b$
z	z
i	k
j	i
k	j

Let ψ be a constrained ordinally efficient and claimwise stable mechanism.

We now claim that under the true preference profile R, $\psi(R, \Delta)_{i,a} = 0$, $\psi(R, \Delta)_{i,b} = 1/6$ and $\psi(R, \Delta)_{i,c} = 5/6$. For ease of notation, we write ψ to denote the outcome.

First, given that object b is the best alternative of agent i and $Pr_\Delta(i \succ_b j) = 1$, due to claimwise stability, we have $\psi_{j,b} = 0$. Given this along with object a being the second choice of agent j, by claimwise stability, we have $\psi_{i,a} = 0$ (as $Pr_\Delta(j \succ_a i) = 1$). Indeed, $\psi_{j,a} = 1$ as he is the highest priority agent with certainty at object a and $\psi_{j,b} = 0$. On the other hand, since $Pr_\Delta(k \succ_b i) = 5/6$ and object b is the favorite one of agent k, $\psi_{i,b} \leq 1/6$.

[44]It is easy to follow from the proof that we indeed only need $0 < \Delta(\succ_b) \leq 1/6$. Hence, in general, the result holds for any priority structure involving the given kind of random ordering and the deterministic structure.

Assume for a contradiction that $\psi_{i,b} < 1/6$. As $Pr_\Delta(i \succ_c z) = 1$ and $\psi_{i,a} = 0$, $\psi_{i,b} < 1/6$ implies that $\psi_{z,c} < 1/6$. On the other hand, since agent z is the top agent at the priority order of object b and ψ is claimwise stable, we have $\psi_{k,b} < 1/6$. These, along with $\psi_{j,b} = 0$, imply that $\psi_{z,b} > 0$. On the other hand, as $\psi_{i,a} = 0$, $\psi_{i,b} < 1/6$, and agent i is the top priority one at object c, we have $\psi_{i,c} > 0$. This allocation, however, cannot be constrained ordinally efficient as agent i can receive arbitrarily small amount of object b from agent z in return of the same amount of object c. As $\psi_{i,b} < 1/6$ (note that agent z has the second highest priority at object c), this trade would not violate claimwise stability (it would have violated if $\psi_{i,b} = 1/6$ due to agent k), and it would improve the welfare of agents i and z. Hence, $\psi_{i,b} = 1/6$.

By claimwise stability, $\psi_{i,b} = 1/6$ and $\psi_{i,a} = 0$ imply that $\psi_{z,c} = 1/6$. We now claim that $\psi_{i,d} = 0$. Assume for a contradiction that $\psi_{i,d} > 0$. In this case, we have $\psi_{k,c} > 0$. Agent i now can give some amount of object d to agent k in return of the same amount of object c. While this trade would not violate claimwise stability, it would improve agents i and z's welfare, which contradicts the constrained ordinal efficiency of ψ. Therefore, $\psi_{i,b} = 1/6$ and $\psi_{i,c} = 5/6$.

Next, consider the false preference relation $R'_i : a, b, c, d$. Let us write ψ' for the outcome of mechanism ψ at (R'_i, R_{-i}, Δ). We claim that $\psi'_{i,a} = 1/6$, $\psi'_{i,b} = 1/6$, and $\psi'_{i,c} = 2/3$. Due to the facts that ψ' is claimwise stable, $Pr_\Delta(k \succ_b i) = 5/6$, and object b is the best alternative of agent k, we have $\psi'_{i,b} \leq 1/6$. In this case, $\psi'_{i,a}$ might be positive if agent j obtains object b with some positive probability (this might be possible here as b is not the top object of agent i with respect to R'_i, and $Pr_\Delta(j \succ_b k) = 1/6$). As $Pr_\Delta(k \succ_b j) = 5/6$, we have $\psi'_{j,b} \leq 1/6$, implying that $\psi'_{i,a} \leq 1/6$.

Now, assume for a contradiction that $\psi'_{i,a} < 1/6$ and $\psi'_{i,b} = 1/6$. Due to the claimwise stability of ψ', they imply that $\psi'_{z,c} \geq 1/6$. On the other hand, due to claimwise stability and $\psi'_{i,a} < 1/6$, we have $\psi'_{j,b} < 1/6$ (note that $Pr_\Delta(i \succ_b j) = 1$). By claimwise stability, we have $\psi'_{j,a} > 0$. These facts altogether show that agent i can get an arbitrarily small amount of object a from agent j in return of the same amount of object b, which would improve their welfare. This trade would not violate claimwise stability (recall that agent i has the second highest priority at object a) as (i) $\psi'_{j,b} < 1/6$, (ii) $\psi'_{z,c} \geq 1/6$, and (iii) $Pr_\Delta(j \succ_b k) = 1/6$, contradicting the constrained ordinal efficiency of ψ'.

Next, assume that $\psi'_{i,a} = 1/6$ and $\psi'_{i,b} < 1/6$. As $\psi'_{i,a} = 1/6$, we have $\psi'_{j,b} = 1/6$. Then, by the claimwise stability of ψ', we have $\psi'_{z,c} < 1/3$. This shows that $\psi'_{k,b} < 1/3$. Hence, $\psi'_{z,b} > 0$. On the other hand, as $\psi'_{i,a} + \psi'_{i,b} < 1/3$, by the claimwise stability of ψ', $\psi'_{l,c} < 1/3$ for each $l \in \{j, k, z\}$. Hence, $\psi'_{i,c} > 0$. In this case, agent i can receive an arbitrarily small amount of object b from agent z in return of the same amount of object c, which would improve the welfare of agents i and z. Moreover, this trade would not violate claimwise stability as $\psi'_{i,b} < 1/6$ (by our supposition), contradicting the constrained ordinal efficiency of ψ'.

The last case is $\psi'_{i,a} < 1/6$ and $\psi'_{i,b} < 1/6$. By claimwise stability, $\psi'_{z,c} < 1/3$. Then, by the same arguments as above, $\psi'_{i,c} > 0$. On the other hand, as $Pr_\Delta(i \succ_b j) = 1$, we have $\psi'_{j,b} < 1/6$. Similarly, as $Pr_\Delta(z \succ_b k) = 1$ and $\psi'_{z,c} < 1/3$, we have $\psi'_{k,b} < 1/3$. Then, these inequalities imply that $\psi'_{z,b} > 0$. In this case, however, agent i can receive arbitrarily small amount of object b from agent z in return of the same among object c. This trade would improve the welfare of them while being compatible with claimwise stability, contradicting the constrained ordinal efficiency of ψ'.

We therefore show that $\psi'_{i,a} = \psi'_{i,b} = 1/6$. Now, we claim that $\psi'_{i,d} = 0$. Assume for a contradiction that $\psi'_{i,d} > 0$. As $\psi'_{z,c} \leq 1/3$, it implies that either (or both) $\psi'_{j,c} > 0$ or $\psi'_{k.c} > 0$. If $\psi'_{k,c} > 0$, then agent i can give some amount of object d to agent k in return of the same amount of object c. While this trade would not violate claimwise stability, it would make agents i and k better off, contradicting the constrained ordinal efficiency of ψ (recall that agents i and k are the top priority ones at objects c and d, respectively). On the other hand, if $\psi'_{j,c} > 0$ and $\psi'_{k,c} = 0$, then we have two cases to consider. First, if $\psi'_{j,d} < \psi'_{k,b}$, then agent i can give arbitrarily small amount of object d to agent j in return of the same amount of object c (recall that agent j has the second highest priority at object d). This would make them better off while not violating claimwise stability. On the other hand, if $\psi'_{j,d} = \psi'_{k,b}$ (the former cannot be greater than the latter due to claimwise stability), then we have $\psi'_{k,a} > 0$ (note that by supposition, $\psi'_{i,d} > 0$ and $\psi'_{k,c} = 0$). If $\psi'_{k,b} = \psi'_{j,d} = 0$, then we have $\psi'_{k,d} = 1$ as he is the top priority agent at object d. This case contradicts our starting supposition that $\psi'_{i,d} > 0$. Let us assume $\psi'_{k,b} = \psi'_{j,d} > 0$. In this case, agent j can give some positive amount of object d to agent k in return of the same amount of object a (recall that we found above that $\psi'_{k,a} > 0$). While this trade is compatible with claimwise stability, it would improve agents k&j's welfare. This, however, contradicts the constrained ordinal efficiency of ψ', showing that $\psi'_{i,d} = 0$. Therefore, $\psi'_{i,a} = \psi'_{i,b} = 1/6$, and $\psi'_{i,c} = 2/3$. It is easy to verify that ψ'_i ordinally dominates ψ_i with respect to R_i, which finishes the proof.

□

Proof of Lemma 1. Let σ be a constrained ordinally efficient matching at problem (R, Δ). Assume for a contradiction that there exist an agent i and objects a, b such that aP_ib, $\sigma_{i,b} > 0$, and $\sum_{k \in N} \sigma_{k,a} < 1$. Then, by constrained ordinal efficiency, we have $\sigma_{i,a} = Pr_\Delta(i \succ_a j) + \sum_{c \in SU(R_j,a)} \sigma_{j,c}$ for some agent j (since otherwise, we can give an arbitrarily small amount of object a to agent i, which would improve the welfare of agent i without violating claimwise stability).[45] As $Pr_\Delta(j \succ_a i) \leq 1$, it shows that $\sigma_{i,a} \geq \sum_{c \in SU(R_j,a)} \sigma_{j,c}$. This, along with our initial supposition $\sum_{k \in N} \sigma_{k,a} < 1$, shows that $\sigma_{j,a} < 1 - \sum_{c \in SU(R_j,a)} \sigma_{j,c}$. This in turn implies that there exists an object d such that aP_jd and $\sigma_{j,d} > 0$. By the same argument as above, it implies that there exists an agent ℓ such that $\sigma_{j,a} = Pr_\Delta(j \succ_a \ell) + \sum_{c \in SU(R_\ell,a)} \sigma_{\ell,c}$. We now claim that $\ell \neq i$. Assume for a contradiction that $\ell = i$. Then, we have $\sigma_{i,a} \geq 1 - Pr_\Delta(j \succ_a i)$ and $\sigma_{j,a} \geq 1 - Pr_\Delta(i \succ_a j)$. These inequalities imply that $\sigma_{i,a} + \sigma_{j,a} \geq 2 - Pr_\Delta(i \succ_a j) - Pr_\Delta(j \succ_a i)$. From here, since $Pr_\Delta(i \succ_a j) = 1 - Pr_\Delta(j \succ_a i)$, we obtain $\sigma_{i,a} + \sigma_{j,a} \geq 1$, which contradicts our very first supposition; hence, $\ell \neq i$.

Next, by the same reasoning as before, there exists an object z such that $aP_\ell z$ and $\sigma_{\ell,z} > 0$. This in turn implies that there exists an agent h such that $\sigma_{\ell,a} = Pr_\Delta(\ell \succ_a h) + \sum_{c \in SU(R_h,a)} \sigma_{h,c}$. Then, by following the same steps in the previous paragraph, we can easily show that $h \neq j$. Furthermore, in what follows, we show that $h \neq i$ as well.

Assume for a contradiction that $h = i$. We already have $\sigma_{i,a} \geq 1 - Pr_\Delta(j \succ_a i)$ and $\sigma_{j,a} \geq 1 - Pr_\Delta(\ell \succ_a j)$. The last finding also implies that $\sigma_{\ell,a} \geq 1 - Pr_\Delta(i \succ_a \ell)$. Now, we can decompose $Pr_\Delta(i \succ_a \ell)$ as follows : $Pr_\Delta(i \succ_a \ell) = \sum_{\succ \in supp(\Delta):\ j \succ_a i \succ_a \ell} \Delta(\succ) + \sum_{\succ \in supp(\Delta):\ i \succ_a \ell \succ_a j} \Delta(\succ) + \sum_{\succ \in supp(\Delta):\ i \succ_a j \succ_a \ell} \Delta(\succ)$. The sum of the last two terms is less than or equal to $Pr_\Delta(i \succ_a j)$. On the other hand, we have $\sigma_{i,a} \geq Pr_{\Delta_a}(i \succ_a j)$. The first term, moreover, is less than or equal to $Pr_\Delta(j \succ_a \ell)$. Similar to above, we also have $\sigma_{j,a} \geq Pr_\Delta(j \succ_a \ell)$. These findings therefore show that $Pr_\Delta(i \succ_a \ell) \leq \sigma_{i,a} + \sigma_{j,a}$. Hence, $\sigma_{\ell,a} \geq 1 - \sigma_{i,a} - \sigma_{j,a}$. This implies that $\sigma_{i,a} + \sigma_{j,a} + \sigma_{\ell,a} \geq 1$, which contradicts our very first supposition. Hence, $h \in N \setminus \{i, j, \ell\}$.

If we continue in the same manner as above, we obtain an object u such that aP_hu and $\sigma_{h,u} > 0$.

[45] As there is no constraint in the allocation of the null object, if $\emptyset P_ib$ and $\sigma_{i,b} > 0$, then this would directly contradict the constrained ordinal efficiency of σ.

This implies that there exists an agent l such that $\sigma_{h,a} = Pr_\Delta(h \succ_a l) + \sum_{c \in SU(R_l,a)} \sigma_{l,c}$. Then, by following the same steps as above, we can easily show that $l \in N \setminus \{i, j, \ell, h\}$.

From our above analysis, agent i cannot consume object a more than $\sigma_{i,a}$ since it would otherwise violate the claimwise stability constraint imposed on agent i due to agent j. We can illustrate this relation by drawing an arrow coming from agent i and going to agent j. If we do the same thing to all other agents as well, we obtain the following figure:

$$i \to j \to \ell \to h \to l.$$

In the above analysis, we prove that each agent is different in the above sequence. Continuing in the same manner would give us a sequence of agents in the above sense where each of them is different. This is, however, impossible as the set of agents is finite. Hence, σ is non-wasteful.

□

Proof of Theorem 3. **"If" Part**: We have already shown that the CPS is claimwise stable and constraint ordinally efficient (we proved it in the absence of the null object; yet the results easily carry over to the presence of the null object case). Hence by Lemma 1, it is non-wasteful. Therefore, it is enough to show that the CPS is binding. To this end, given any problem instance (R, Δ), consider an agent-object pair (i, a). In the course the CPS, if he stops consuming object a at the time it is totally exhausted, then there is no agent k such that $F(R_i, a, CPS_i) < F(R_k, a, CPS_k)$ and $CPS_{k,a} > 0$. Otherwise, by the definition of the CPS, there has to be an agent k who applies to object a before agent i stops consuming it (note that this is due the definition of Θ (see the formal description of the CPS)), and $CPS_{i,a} = Pr_\Delta(i \succ_a k) + \sum_{c \in SU(R_k,a)} CPS_{k,c}$. Therefore, the CPS is binding.

"Only If" Part: Let us consider a mechanism ψ which is non-wasteful, claimwise stable, and binding. We use the same construction introduced in the proof of Theorem 1. For the sake of completeness, we repeat it here. First, for ease of notation, we suppress the dependency of the mechanisms ψ and CPS on the problem instance (R, Δ) and just write ψ and CPS to denote their outcomes at (R, Δ). Let $\pi_1 = min\{F(R_i, a, \psi_i) : \text{for all } (i, a) \in N \times O\}$, $\pi_k = min\{F(R_i, a, \psi_i) : F(R_i, a, \psi_i) > \pi_{k-1} \text{ for all } (i, a) \in N \times O\}$. We also write $\Pi = \{F(R_i, a, \psi_i) : \text{for all } (i, a) \in N \times O\}$.

We now claim that $F(R_i, a, \psi_i) = F(R_i, a, CPS_i)$ for any agent-object pair (i, a), which will in turn imply that $\psi = CPS$. For this purpose, we inductively prove that $F(R_i, a, \psi_i) = F(R_i, a, CPS_i)$ for all agent-object pair (i, a) such that $F(R_i, a, \psi_i) = \pi_k$.

Let $k = 1$ and $F(R_i, a, \psi_i) = \pi_1$. First, by the claimwise stability of ψ and the definition of the CPS, for any agent-object pair (j, b) with $F(R_j, b, \psi_j) = \pi_1$, we have $F(R_j, b, CPS_j) \geq \pi_1$. Hence, in particular, $F(R_i, a, CPS_i) \geq \pi_1$. Let us first assume that object a is the top choice of agent i. If $\pi_1 = 1$, then we have $F(R_i, a, \psi_i) = F(R_i, a, CPS_i)$. Assume that $\pi_1 < 1$. Then, we have two cases to consider.

Case 1. There exists no agent h such that $F(R_i, a, \psi_i) < F(R_h, a, \psi_h)$ and $\psi_{h,a} > 0$. By non-wastefulness, it implies that object a is totally exhausted at time π_1 in the course of the consumption process associated to ψ. This, along with the fact that $F(R_j, b, \psi_j) \leq F(R_j, b, CPS_j)$ for any agent-object pair (j, b) such that $F(R_j, b, \psi_j) = \pi_1$, shows that $F(R_i, a, CPS_i) = F(R_i, a, \psi_i) = \pi_1$.

Case 2. Let us assume that there exists an agent h such that $F(R_i, a, \psi_i) < F(R_h, a, \psi_h)$ and $\psi_{h,a} > 0$. This means that agent i stops consuming object a before it is totally exhausted. As ψ is binding, there exists an agent k who applies to object a in a step before the one in which agent i stops consuming object a. If we write $s(k)$ and $s(i)$ to respectively denote those steps, then we have $s(k) \prec s(i)$. and $\psi_{i,a} = Pr_\Delta(i \succ_a k) + \sum_{c \in SU(R_k, a)} \psi_{k,c}$.

If a is the best object of agent k, then the above binding claimwise stability constraint implies that $Pr_\Delta(k \succ_a i) = 1 - \pi_1$. This is because of the fact that $\sum_{c \in SU(R_k, a)} \phi_{k,c} = 0$.[46] Hence, under any claimwise stable mechanism ϕ, we have $\phi_{i,a} \leq \pi_1$. This, along with $F(R_i, a, CPS_i) \geq \pi_1$, implies that $F(R_i, a, CPS_i) = \pi_1$.

Let us now consider the case where $SU(R_k, a) \neq \emptyset$. The above binding claimwise stability constraint now implies that $Pr_\Delta(k \succ_a i) = 1$ and $\sum_{c \in SU(R_k, a)} \psi_{k,c} = \pi_1$ (recall that by our construction, $\sum_{c \in SU(R_k, a)} \psi_{k,c} \geq \pi_1$). Let object b be the just preferred object to object a by agent k. By construction, we have $F(R_k, b, \psi_k) = \pi_1$. This case, therefore, gives another agent-object pair (k, b) such that $F(R_k, b, \psi_k) = \pi_1$, which is similar to our starting pair (i, a).

We now repeat the above analysis for agent k. If there is no object $c \in U(R_k, b)$ such that Case 2 applies to it, then by the same arguments, we have $F(R_k, b, CPS_k) = F(R_k, b, \psi_k) = \pi_1$. This in

[46] As object a is the top alternative of agent k.

turn implies that $CPS_{i,a} = \pi_1$ due to claimwise stability; hence $F(R_i, a, CPS_i) = \pi_1$.

Let us assume that there exists an object $c \in U(R_k, b)$ such that Case 2 applies to it. This implies that there exists an agent j applying to object c in a step before the one in which agent k stops consuming it in the consumption process associated to ψ. Let $s(j)$ denote that step. Note that $s(j) \prec s(k) \prec s(i)$ (recall that $s(k)$ is the step in which agent k applies to object a, which is dispreferred to object c by himself). Moreover, we have $\psi_{k,c} = Pr_{\Delta}(k \succ_c j) + \sum_{d \in SU(R_j,c)} \psi_{j,d}$.

If we apply the above steps to agent j, we will either directly show that $F(R_k, b, CPS_k) = \pi_1$ (by following the same arguments as above) or find another agent-object pair (h, d) where $d \in U(R_j, c)$, and $\psi_{j,d} = Pr_{\Delta}(j \succ_d h) + \sum_{e \in SU(R_h,d)} \psi_{h,e}$. If the former is the case, then $F(R_k, b, CPS_k) = \pi_1$ implies that $F(R_i, a, CPS_i) = \pi_1$ (by claimwise stability of the CPS). On the other hand, if the latter is the case, then we obtain agent h such that $\psi_{j,d} = Pr_{\Delta}(j \succ_d h) + \sum_{e \in SU(R_h,d)} \psi_{h,e}$. As ψ is binding, step $s(h)$ in which agent h applies to object d occurs before step $s(j)$. That is, we have $s(h) \prec s(j) \prec s(k) \prec s(i)$.

If we continue in the same manner, there are two cases. We may find an agent such that Case 2 does not apply to him for any corresponding object. In this case, similar to above, $F(R_i, a, CPS_i) = \pi_1$ would follow from the claimwise stability of the CPS. Otherwise, we add another step to the above sequence of the steps. However, since everything is finite and $\prec$ is transitive (by its definition), this case cannot happen all the time, showing that there has to be agent-object pair to which Case 2 does not apply. Hence, $F(R_i, a, CPS_i) = \pi_1$.

So far, we assume that object a is the best alternative of agent i. Let us suppose that it is his second best object. Let object b is his first choice. Then, we have $F(R_i, b, \psi_i) = F(R_i, a, \psi_i) = \pi_1$, implying that $\psi_{i,a} = 0$. Moreover, by our above analysis, we have $F(R_i, b, \psi_i) = F(R_i, b, CPS_i) = \pi_1$. As $\psi_{i,a} = 0$ and $\pi_1 < 1$, there exists an agent j such that $Pr_{\Delta}(j \succ_a i) = 1$ and $\sum_{c \in SU(R_j,a)} \psi_{j,c} = 0$ (as ψ is binding). Note that this implies that $\pi_1 = 0$. If a is the top choice of agent j, then the result follows from the claimwise stability of the CPS. Let us assume that object a is not the top choice of agent j. Then, as $\psi_{i,a} = 0$, Case 2 above applies to the pair (i, a). Once we repeat the same arguments in Case 2 (the arguments there do not depend on the ranking of object a at agent i's preference list), we conclude that $\psi_{i,a} = 0$. This fact, along with $F(R_i, b, CPS_i) = \pi_1$, shows that $F(R_i, a, CPS_i) = \pi_1$. The other cases of object a not being the top two objects of agent i

follow from the same arguments, finishing the proof of the base step of the induction.

For the induction hypothesis, assume that $F(R_i, a, \psi_i) = F(R_i, a, CPS_i)$ for all $F(R_i, a, \psi_i) = \pi_k$ where $k < k'$. Let $F(R_i, a, \psi_i) = \pi_{k'}$. We want to show that $F(R_i, a, CPS_i) = \pi_{k'}$.

First, due to the (i) induction hypothesis, (ii) the definition of the CPS, and (iii) the claimwise stability of ψ, we have $F(R_i, a, CPS_i) \geq \pi_{k'}$.

Similar to above, if there is no agent k such that $F(R_i, a, \psi_i) < F(R_k, a, \psi_k)$ and $\psi_{k,a} > 0$, then this implies that agent i continues to consume object a until it is totally exhausted in the course of the consumption process associated to ψ. In this case, by the same previous arguments, $F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'}$, which finishes the proof.

For the other case, let us now assume that there exists an agent k such that $F(R_i, a, \psi_i) < F(R_k, a, \psi_k)$ and $\psi_{k,a} > 0$. As ψ is binding, there has to be an agent j (may be the same as agent k) such that $\psi_{i,a} = Pr_{\Delta}(i \succ_a j) + \sum_{c \in SU(R_j, a)} \psi_{j,c}$. Moreover, agent j applies to object a in a step before the one in which agent i stops consuming object a.

Assume that $\psi_{i,a} > 0$. Let object b be the object just preferred to object a by agent j (if object a is the best alternative of agent j, then the proof follows from the claimwise stability of the CPS and the above binding claimwise stability constraint). As agent j applies to object a before agent i stops consuming it, we have $F(R_j, b, \psi_j) \leq \pi_{k'}$. If it holds strictly, then by the induction hypothesis, we have $F(R_j, b, \psi_j) = F(R_j, b, CPS_j) < \pi_{k'}$. Then, by the claimwise stability of the CPS, we have $CPS_{i,a} = \psi_{i,a}$. This finding with $\psi_{i,a} > 0$ and the induction hypothesis show that $F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'}$. For the other case, assume that $F(R_j, b, \psi_j) = \pi_{k'}$. In this case, similar to the previous arguments, we apply the same steps to agent j. In the course of this iterative process, there are two cases to consider. First, we may find an agent h and the corresponding object c such that $F(R_h, c, \psi_h) < \pi_{k'}$ (here, object c is the one just preferred to object b by agent h). In this case, as the same as above, the proof would follow from the claimwise stability of the CPS and the induction hypothesis. On the other hand, if we may not find such an agent-object pair, then it would give us a cycle in the ordering of the steps as before, which would constitute a contradiction. This, hence, shows that there exists an agent-object pair falling into the first case, showing that $CPS_{i,a} = \psi_{i,a}$. This finding with $\psi_{i,a} > 0$ and the induction hypothesis show that $F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'}$.

If $\psi_{i,a} = 0$, then let object b be such that $F(R_i, b, \psi_i) = \pi_{k'}$ and $\psi_{i,b} > 0$. This implies that bP_ia. By the above analysis, we have $F(R_i, b, \psi_i) = F(R_i, b, CPS_i)$. This implies that $\psi_{i,c} = 0$ for all $c \in U(R_i, a) \backslash U(R_i, b)$. Our binding claimwise stability constraint $\psi_{i,a} = Pr_\Delta(i \succ_a j) + \sum_{c \in SU(R_j,a)} \psi_{j,c}$, along with $\psi_{i,a} = 0$, implies that $Pr_\Delta(j \succ_a i) = 1$ and $\sum_{c \in SU(R_j,a)} \psi_{j,c} = 0$. By the first step of the induction, we have $\sum_{c \in SU(R_j,a)} CPS_{j,c} = 0$ as well. Hence, by the claimwise stability of the CPS, we have $CPS_{i,a} = 0$. The same arguments can be directly applied to any object $c \in U(R_i, a) \setminus U(R_i, b)$ to demonstrate that $CPS_{i,c} = 0$ for each of such object c (recall that, for any $c \in U(R_i, a) \setminus U(R_i, b)$, $\psi_{i,c} = 0$). Therefore, $F(R_i, a, \psi_i) = F(R_i, a, CPS_i) = \pi_{k'}$.

□

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