Ex-ante versus Ex-post Proportional Rules for State Contingent Claims*

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Abstract

We consider rationing problems where the claims are state contingent, i.e., each agent submits a claim for every state in stage one and the realization of state happens in stage two. A rule must distribute the resources in the first stage before the realization of the state of the world. We introduce two natural extensions of proportional rules in this framework, namely, ex-ante and ex-post proportional rules, and characterize them using axioms standard in literature.

Keywords Rationing · State contingent claims · No advantageous reallocation · ex-ante proportional rule · ex-post proportional rule

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1 Introduction

Rationing problem is arguably the simplest model of the distributive justice. The problem involves a resource to be divided among a number of agents who submit claims for the resource. Rationing is required when the sum of the claims is larger than the resource, with typical examples being bankruptcy, taxation, inheritance, etc. Perhaps the problem of rationing is as old as the history of civilization itself and one can find documentation of such problems in ancient texts such as Talmud and Aristotle's books. The very first formal analysis of the rationing problem was presented by O'Neill (1982) where he interprets the resource as inheritance. The problem of rationing concerns more of ethical or normative issues since market or traditional institutions can not convincingly provide a way out. For this reason adopting axiomatic approach has been the focus of the literature on rationing. Probably the most natural rule in this context arises from Aristotle's maxim, "Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences" from Nicomachean Ethics. *Proportional* rule gives shares in proportion to claims. There are various normative treatments of the Proportional rule, such as O'Neill (1982), Moulin (1987), Chun (1988), Young (1988), and Ju et al. (2007), etc.

Other rules central in the literature consider normative axioms including various forms of egalitarianism. *Uniform Gains* rule equalizes the shares such as the shares do not exceed the claims. *Uniform Losses* rule equalizes the losses (difference between claim and share) as much as possible. One can refer to some axiomatic characterizations of egalitarian rules for different environments in Dagan (1996), Herrero and Villar (2001), Sprumont (1991), Juarez and Kumar (2013) etc. Young (1987a) characterizes a class of parametric rules in the taxation problem and Young (1987b) introduces another interesting family of rules called the equal sacrifice rules. Rules from ancient texts and their extensions have also been considered by various authors. Aumann and Maschler (1985) provides a rule from Talmud in the bankruptcy context and papers like, Hokari and Thomson (2003) studies generalizations of the same. Alcalde, Marco, and Silva (2005) extends an old solution for bankruptcy problems described by Ibn Ezra in the XII century. One can see Moulin (2002) and Thomson (2003) for survey of axiomatic characterization of the rationing rules.

We consider the rationing problems in a two stage setting where the claims are state contingent. In stage one, each agent submits a claim for every possible state of the world. The realization of state happens in stage two. A rule must distribute the resources in the stage one i.e., before the realization of the state of the world. Such a situation may arise, for instance, in the allocation of fiscal budget of a country. Different departments of the government may require different amounts based on the state of the world to be realized in the coming fiscal year. For example, the Department of Defense may have different requirements depending on its relations with the neighboring countries in the following year. Department of Agriculture have requirements based on factors like the rainfall in the following year. The Department of Health may have requirements that depend on factors like incidence of epidemic, weather, etc. However, the federal budget must be allocated at the beginning of the fiscal year. Another example of our setting is the distribution of research funds (or travel grants) among the graduate students of a department in a university who expect travels or research expenses based on the state of the world (e.g., expenses based on the results of her research, travels based on the conferences accepting her paper, etc.). A situation like our setting also arises in the allocation of university funds among different departments based on their performance/need, or NSF funds to researchers from various universities, etc.

This natural framework of rationing problem has not been given much consideration in the literature. A fairly close setting called multi-issue allocations (MIA), introduced in Calleja et al. (2005), has been studied in the literature. Bergantiños et al. (2011), and Lorenzo et al. (2009) provide several axiomatic characterizations of Uniform Gains and Uniform Losses rules in MIA whereas Moreno-Ternero (2009), and Bergantiños et al. (2010) provide axiomatic characterizations of Proportional rule in MIA. The MIA framework does not consider uncertainty. A similar framework to ours that considers uncertainty has been studied by Habis and Herings (2013). They are interested in checking the stability[1] of the stochastic extensions of various rationing rules and show that the only stable rule is the stochastic extension of the Uniform Gains.

In our two-stage framework where the agents submit their claims in the first stage and uncertainty is resolved in the second stage, the resource must be allocated in the first stage. Two particularly natural approaches in such situations arise. The first one is to apply a rationing rule on the expectation of

[1] They used "Weak Sequential Core" as the stability criterion which was defined in Habis and Herings (2011).

the claims, which we call *ex-ante* rationing rule. The other approach is to consider the expectation of the allocations (by a rule) corresponding to the various state contingent claims, which we call *ex-post* rationing rule. In this paper, we will focus our attention to proportional rules and characterize both ex-ante and ex-post proportional rules. Our axiomatic characterizations are based on *No Advantageous Reallocation* axiom introduced by Moulin (1985). This axiom states that no group of agents can benefit from reallocating their claims amongst them. We extend this concept to our state contingent framework and introduce two nonmanipulability conditions. The first extension which we call *No Advantageous Reallocation across Individuals* (*NARAI*) requires that no group of agents benefits if transfers are allowed within a state. The next extension considers transfers across states which we call *No Advantageous Reallocation across States* (*NARAS*). We also use the axioms of *Anonymity* (*AN*), *Symmetry* (*SYM*), *Continuity* (*CONT*), *No Award for Null* (*NAN*), and *Independence* (*IND*). *AN* says that the rule should not distinguish based on the names of the agents and *SYM* requires that the names of the states do not matter. *CONT* states that the rule should be a continuous function in its arguments and *NAN* says that agents with zero claims in all states should be allocated zero amount. *IND* says that if we mix two lotteries[2] with a third one, then the rationing rule associated with these two mixed lotteries does not depend on the third lottery used. We show that ex-ante proportional rule is the only rule satisfying *AN*, *SYM*, *CONT*, *NAN*, *NARAI*, and *NARAS* whereas ex-post proportional rule is characterized by *AN*, *SYM*, *CONT*, *NAN*, *NARAI*, and *IND*.

Another interesting aspect of this problem is to compare the shares allocated by the ex-ante and ex-post proportional rules. In the appendix, we do the comparison for the ex-ante and ex-post allocations by the proportional rules for various distributions of claims and find sufficient conditions under which a particular agent will be favoured by one approach compared to the other. Section 2 provides the preliminaries. Section 3 gives the characterization results, and section 4 concludes with some directions for future research.

[2] By lottery we mean probability distribution over states of the world to be realized in the stage two.

2 Preliminaries

In the state contingent claims' framework, a rationing problem is a tuple (N, S, x, p, t) where N is a finite set of agents and S is a finite set of the states of the world.[3] The state contingent claim matrix $x \in \mathbb{R}_+^{N\times S}$ represents the claims of agents in various states, where x_{is} denotes the claim of agent i in state s. The probabilities of states is denoted by $p \in \Delta^{|S|-1}$ and $t \geq 0$ is the resource to be shared among the agents.[4] It is assumed that $\sum\limits_{i\in N} x_{is} \geq t$ for all $s \in S$. Throughout the paper, we consider a fixed population N and a fixed set S of states. For the sake of brevity, we denote our problem (x, p, t). A non-empty set of problems is called a domain and is denoted by $\mathcal{D}$.[5] A rationing rule $\varphi : \mathcal{D} \to \mathbb{R}_+^N$ gives a vector of shares such that $\sum\limits_{i\in N} \varphi_i(x, p, t) = t$. We restrict our attention to rich domains which is defined as follows:

Definition 1 *A domain $\mathcal{D}$ is* rich *if for all $x, x' \in \mathbb{R}_+^{N\times S}$ for all $p \in \Delta^{|S|-1}$ for all $t \geq 0$ with $x_{Ns} = x'_{Ns}$[6] for all $s \in S$, then $\{(x, p, t) \in \mathcal{D} \Rightarrow (x', p, t) \in \mathcal{D}\}$.*

Now we will define two rationing rules which involve proportionality idea. Since our rules are based on proportionality idea, let us recall the standard proportional rule when there is only one (certain) state of the world s, i.e., $S = \{s\}$.

The proportional rule is defined as

$$pr_i(x, p, t) = \frac{x_{is}}{x_{Ns}} t \text{ for all } i \in N$$

[3] The standard rationing problem is defined as (N, x, t) where N is a finite set of agents, x is a claim vector $x = (x_i)_{i\in N} \geq 0$ and $t \geq 0$ is the resource to be shared among the agents. A rationing rule φ, assigns a vector of shares $\varphi(N, x, t) \in \mathbb{R}_+^N$ to every rationing problem such that $\sum\limits_{i\in N} \varphi_i(N, x, t) = t$.

[4] $\Delta^{|S|-1}$ denotes a $|S| - 1$ dimensional simplex.

[5] More precisely this is a restricted domain of problems where N and S are fixed so a better notation would be $\mathcal{D}(N, S)$. However, for notational simplicity we use $\mathcal{D}$ since it does not raise any confusion.

[6] We use the notation $x_{Ts} := \sum_{i\in T}(x_{is})$, where $T \subseteq N$.

Ex-ante proportional rule is defined as applying proportional rule to the expectation of the state contingent claims.

$$\overline{pr_i}(x,p,t) := pr_i\left(E_p\left[x\right],t\right) = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{j\in N}\sum\limits_{s\in S}(p_s x_{js})}t = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{Ns})}t \text{ for all } i\in N$$

Ex-post proportional rule is defined as expectation of the shares found by the proportional rule on the state contingent claims.

$$\widetilde{pr}_i(x,p,t) := E_p\left[pr_i\left(x,t\right)\right] = \sum_{s\in S}\left(p_s\frac{x_{is}}{x_{Ns}}\right)t \text{ for all } i\in N$$

The illustration of the ex-ante and ex-post proportional rules for a simple economy with two people and two states is presented in Figure 1.

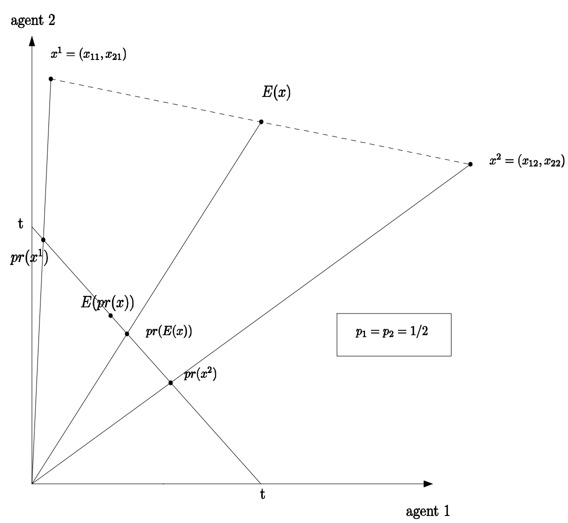

Figure 1: Ex-ante vs Ex-Post Proportional Rule

As it is shown in the Figure 1, the ex-ante and ex-post proportional rules do not necessarily coincide. For $|N| = 2$, aforementioned rules give identical shares when the sum of the claims is equal for each state (Figure 2) or the ratio of the claims for each state is equal (Figure 3). The difference of the shares for the ex-ante and ex-post proportional rules for general economy with finite number of people and states are given in the Proposition 1 in the Appendix.

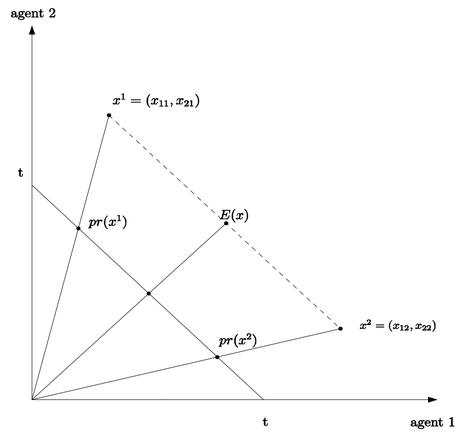

Figure 2: The sum of the claims for each state is equal: $x_{N1} = x_{N2}$

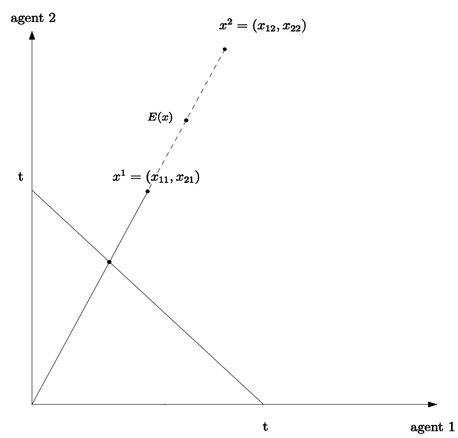

Figure 3: The ratio of the claims for each state is equal: $\frac{x_{11}}{x_{21}} = \frac{x_{12}}{x_{22}}$

3 Characterizations

In this section we introduce some axioms that will be used to characterize the rules we have discussed above.

Continuity (CONT): For all $(x, p, t) \in \mathcal{D}$ and for all sequences $(x^k, p^k, t^k) \in \mathcal{D}$, if $(x^k, p^k, t^k) \to (x, p, t)$, then $\varphi(x^k, p^k, t^k) \to \varphi(x, p, t)$.

Continuity tells us that small changes in the parameters of the problem do not bring big jumps in the allocations. Continuity is desirable because we do not want small errors (e.g., measurement errors) to lead to big changes in the allocations.

Anonymity (AN): For all $(x, p, t) \in \mathcal{D}$, for all permutations $\sigma : N \to N$, and for all $i \in N$, $\varphi_i(x, p, t) = \varphi_{\sigma(i)}(x_\sigma, p, t)$, where $x_\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, ...x_{\sigma(|N|)})$.

Anonymity says that the names of the agents do not matter. This is very natural axiom and is central to the literature on fairness.

Symmetry (SYM): For all $(x, p, t) \in \mathcal{D}$, for all permutations $\rho : S \to S$, and for all $i \in N$, $\varphi_i(x, p, t) = \varphi_i(x_\rho, p_\rho, t)$, where $p_\rho = (p_{\rho(1)}, p_{\rho(2)}, ..., p_{\rho(|S|)})$ and $x_\rho = (x_{\rho(1)}, x_{\rho(2)}, ..., x_{\rho(|S|)})$.

Symmetry is similar to the Anonymity axiom with the role of agents substituted by states. It says that the names of the states do not matter.

No Award for Null (NAN): For all $(x, p, t) \in \mathcal{D}$ and for all $i \in N$, if $x_{is} = 0$ for all $s \in S$, then $\varphi_i(x, p, t) = 0$.

No Award for Null axiom says that an agent with zero claim for each state should get zero share. This axiom is also called dummy axiom in the literature.

Moulin (1985) defined Non-Advantageous Reallocation axiom to characterize the egalitarian and utilitarian solutions in quasi-linear social choice problems. We will define two axioms on invariance to reallocation in a similar manner where transfers are made either across individuals or across states.

Non-advantageous Reallocation across Individuals (NARAI): For all $(x, p, t), (x', p, t) \in \mathcal{D}$ and for all $i \in N$, if $\sum_{j \in N \setminus \{i\}} x_{js} = \sum_{j \in N \setminus \{i\}} x'_{js}$ and $x_{is} = x'_{is}$ for all $s \in S$, then $\varphi_i(x, p, t) = \varphi_i(x', p, t)$.

NARAI states that the share of agent i depends on the sum of the total claim of the agents other than himself. In other words, individuals other than i cannot affect the share of i by reallocating their claims among themselves, i.e. the share of individual i is a function of $x_i, x_{N \setminus i}, p$, and t.

Non-advantageous Reallocation across States (NARAS): For all $(x, p, t), (x', p, t) \in \mathcal{D}$ and for all $i \in N$, if $\sum_{s \in S} (p_s x_{is}) = \sum_{s \in S} (p_s x'_{is})$ and $x_{js} = x'_{js}$ for all $j \in N \backslash \{i\}$ and for all $s \in S$, then $\varphi_j(x, p, t) = \varphi_j(x', p, t)$ for all $j \in N \backslash \{i\}$.

NARAS implies that if agent i reallocates his claim across all the states given his expected claim is constant then the share of the other individuals would not change.

Now we will characterize the class of rules satisfying NARAI, Anonymity, and Continuity.

Theorem 1 *Let $|N| \geq 3$. Let (x, p, t) and $(x', p, t) \in \mathcal{D}$. A rationing rule φ satisfies NARAI, AN, and CONT if there exists a continuous $W_s : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ for all $s \in S$ such that for all $i \in N$ we have*

$$\varphi_i(x, p, t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N, p, t) \right] \quad (1)$$

Conversely every rule satisfying NARAI, AN, and CONT must be in the form of (1).

Proof. The first statement is obvious. We will prove the second statement. Let $(x, p, t) \in \mathcal{D}$. Let φ be a rationing rule satisfying NARAI, AN, and CONT.
Let $x' = (x_1 + x_2, 0, x_3, ...)$. Apply NARAI for the coalition $\{1, 2\}$, we get

$$\varphi_1(x, p, t) + \varphi_2(x, p, t) = \varphi_1(x', p, t) + \varphi_2(x', p, t). \quad (2)$$

Let $x^{''} = (x_1, x_{N \backslash \{1\}}, 0, 0, ...)$. Now we'll apply NARAI for the coalition $N \backslash \{1\}$. This implies

$$\varphi_{N \backslash \{1\}}(x, p, t) = \varphi_{N \backslash \{1\}}(x^{''}, p, t). \quad (3)$$

Thus we have $t - \varphi_{N \backslash \{1\}}(x, p, t) = \varphi_1(x, p, t) = \varphi_1(x^{''}, p, t) = t - \varphi_{N \backslash \{1\}}(x^{''}, p, t)$.
$\varphi_2(x, p, t) = \varphi_1(x_2, x_{N \backslash \{2\}}, 0, ..., 0, p, t)$ by (3) and AN.
$\varphi_1(x', p, t) = \varphi_1(x_1 + x_2, x_{N \backslash \{1,2\}}, 0, ..., 0, p, t)$ by (3).
$\varphi_2(x', p, t) = \varphi_1(0, x_N, 0, ..., 0, p, t)$ by (3) and AN.
Let us plug these back into (2).
$\varphi_1(x_1, x_{N \backslash \{1\}}, 0, ..., 0, p, t) + \varphi_1(x_2, x_{N \backslash \{2\}}, 0, ..., 0, p, t) = \varphi_1(x_1 + x_2, x_{N \backslash \{1,2\}}, 0, ..., 0, p, t) + \varphi_1(0, x_N, 0, ..., 0, p, t)$

Let us define $f : \mathbb{R}^S \times \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}$ such that
$f(x_i, x_N, p, t) = \varphi_1(x_i, x_{N\setminus\{i\}}, 0, ..., 0, p, t) - \varphi_1(0, x_N, 0, ..., 0, p, t)$
and define $g : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}$ such that $g(x_N, p, t) = \varphi_1(0, x_N, 0, ..., 0, p, t)$.
Thus we get

$$f(x_1, x_N, p, t) + f(x_2, x_N, p, t) = f(x_1 + x_2, x_N, p, t).$$

f is additive in the first term and by definition, f is continuous (φ is continuous). Fix (x_N, p, t). So by invoking Cor 3.1.9, p.51, from Eichhorn (1978), we deduce that f is linear in the first term, that is, there exists a continuous $W : \mathbb{R}^S \times \Delta^{|S|-1} \times \mathbb{R}_+ \to \mathbb{R}^S$ such that .
$f(x_i, x_N, p, t) = \sum\limits_{s\in S} [W_s(x_N, p, t)x_{is}]$.
So $\varphi_i(x, p, t) = \sum\limits_{s\in S} [(W_s(x_N, p, t)x_{is})] + g(x_N, p, t)$.
Summing over $i \in N$ we get
$\sum\limits_{i\in N} \varphi_i(x, p, t) = \sum\limits_{s\in S} [(W_s(x_N, p, t)x_{Ns})] + |N|g(x_N, p, t) = t$.
So $g(x_N, p, t) = \frac{t - \sum\limits_{s\in S} [(W_s(x_N,p,t)x_{Ns})]}{|N|}$.
Hence we get the desired functional form.
$\varphi_i(x, p, t) = \frac{t}{|N|} + \sum\limits_{s\in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N, p, t) \right]$ for all $i \in N$. ■

The family of rules characterized in the theorem above contains various rules including proportional and egalitarian rules. We provide some notable rules that belong to this family in the example below.

Example 1 *Various weight functions $W_s(x_N, p, t)$ give rise to various rules. Some of the examples are:*

- We have equal split rule, $\varphi_i(x, p, t) = \frac{t}{|N|}$ when $W_s(x_N, p, t) = 0$.
- When $W_s(x_N, p, t)$ satisfies $\sum\limits_{s\in S} [W_s(x_N, p, t)x_{Ns}] = t$, we get the family of proportional rules, i.e., $\varphi_i(x, p, t) = \sum\limits_{s\in S} [W_s(x_N, p, t)x_{is}]$.
- If the weight functions are uniform with respect to states, that is, $W_s(x_N, p, t) = \frac{t}{\sum\limits_{s\in S} x_{Ns}}$ for all s, then $\varphi_i(x, p, t) = \frac{\sum\limits_{s\in S} x_{is}}{\sum\limits_{s\in S} x_{Ns}} t$.
- We have ex-ante proportional rule, $\varphi_i(x, p, t) = \frac{\sum\limits_{s\in S} (p_s x_{is})}{\sum\limits_{s\in S} (p_s x_{Ns})} t$ when $W_s(x_N, p, t) = \frac{p_s t}{\sum\limits_{s\in S} (p_s x_{Ns})}$.

- We get ex-post proportional rule, $\varphi_i(x,p,t) = \sum\limits_{s\in S}\left(p_s \frac{x_{is}}{x_{Ns}}\right) t$ when $W_s(x_N,p,t) = \frac{p_s t}{x_{Ns}}$.
- The family contains non-symmetric proportional rules with respect to states as well, e.g., $\varphi_i(x,p,t) = \frac{x_{i1}}{x_{N1}} t$ when $W_1(x_N,p,t) = \frac{t}{x_{N1}}$, $W_2(x_N,p,t) = W_3(x_N,p,t) = ... = 0$ (all the weight is given to state 1).

In Theorem 1 we characterized the family of rules which include both ex-ante and ex-post proportional rules. We provide characterization of these rules in the following theorems.

Theorem 2 *Let $|N| \geq 3$. A rationing rule φ satisfies NARAI, NARAS, AN, SYM, CONT, and NAN if and only if φ is ex-ante proportional rule.*

Proof. "If" part is obvious. We will prove the "only if" part. Let $(x,p,t) \in \mathcal{D}$. Let φ be a rationing rule satisfying NARAI, NARAS, AN, CONT, and NAN. Given that φ satisfies the premises of Theorem 1, we have $\varphi_i(x,p,t) = \frac{t}{|N|} + \sum\limits_{s\in S}\left[\left(x_{is} - \frac{x_{Ns}}{|N|}\right) W_s(x_N,p,t)\right]$. Fix $i \in N$ and let $x_{is} = 0$ for all $s \in S$. NAN implies that $\varphi_i(x,p,t) = \frac{t}{|N|} + \sum\limits_{s\in S}\left[\left(x_{is} - \frac{x_{Ns}}{|N|}\right) W_s(x_N,p,t)\right] = 0$. So we get $\sum\limits_{s\in S}\left[W_s(x_N,p,t)x_{Ns}\right] = t$. Hence $\varphi_i(x,p,t) = \sum\limits_{s\in S}\left[(W_s(x_N,p,t)x_{is})\right]$.
Let $\sum\limits_{s\in S}(p_s x_{is}) = \sum\limits_{s\in S}(p_s x'_{is})$ and $x_{js} = x'_{js}$ for all $j \in N\backslash\{i\}$ and for all $s \in S$. So we get

$$\sum_{s\in S}\left[p_s\left(x_{is} - x'_{is}\right)\right] = 0 \text{ for all } i \in N. \tag{4}$$

By NARAS, we have $\varphi_i(x,p,t) = \varphi_i(x',p,t)$ for all $i \in N$. Then

$$\sum_{s\in S}\left[(W_s(x_N,p,t)x_{is})\right] = \sum_{s\in S}\left[(W_s(x'_N,p,t)x'_{is})\right] \text{ for all } i \in N. \tag{5}$$

Fix $j \in N\backslash\{i\}$, by (5) we have $\sum\limits_{s\in S}\left[(W_s(x_N,p,t)x_{js})\right] = \sum\limits_{s\in S}\left[\left(W_s(x'_N,p,t)x'_{js}\right)\right] = \sum\limits_{s\in S}\left[(W_s(x'_N,p,t)x_{js})\right]$.
By the richness of $\mathcal{D}$, we have $W_s(x_N,p,t) = W_s(x'_N,p,t)$ for all $s \in S$. By using (5) we get

$$\sum_{s\in S}\left[W_s(x_N,p,t)\left(x_{is}-x'_{is}\right)\right]=0 \text{ for all } i\in N. \tag{6}$$

By (4) and (6), we deduce that p and W are colinear. So for all $s \in S$ there exists $h_s : \mathbb{R}^S \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $W_s(x_N, p, t) = h_s(x_N, t)p_s$ for all $s \in S$. By SYM, we have $h_s = h_t$ for all $t \in S\backslash\{s\}$. So we can write it as $h(x_N, t)$.
Summing over $i \in N$, $\sum\limits_{i\in N}\sum\limits_{s\in S}[(W_s(x_N,p,t)x_{is})] = \sum\limits_{i\in N}\sum\limits_{s\in S}[h(x_N,t)p_s x_{is}] = h(x_N,t)\sum\limits_{s\in S}(p_s x_{Ns}) = t$.

So $h(x_N,t) = \frac{t}{\sum\limits_{s\in S}(p_s x_{Ns})}$. And $\varphi_i(x,p,t) = \sum\limits_{s\in S}[h(x_N,t)p_s x_{is}] = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{Ns})}t$ for all $i \in N$. ■

Before characterizing ex-post proportional rule let us note that NARAS axiom is not satisfied by ex-post proportional rules. Figure 4 below illustrates an instance of this fact with a two-agent and two-state example where both states are equally likely. Here, agent 2 has a deterministic claim, i.e., he claims c_2 in both states. Now, if agent 1 also has a deterministic claim of, say c_1, in both states then the final allocation by ex-post rule is given by $pr(E(x))$. On the contrary, suppose agent 1 reallocates his claims across the two states (as x_{11} and x_{12}) in such a way that the mean of the claims is preserved at c_1. In this case, the final allocation by the ex-post proportional rule is given by $E(pr(x))$. Clearly, agent 1 is worse off by this mean-preserving spread and thus NARAS is not satisfied.

Now we will characterize ex-post proportional rule. The functional form of ex-post proportional rule exhibits additively separable preferences with respect to the states. Hence we are going to utilize a version of Expected Utility Theorem. To arrive there we are going to utilize an independence axiom in the spirit of von Neumann - Morgenstern.

Independence (IND): For all $(x,p,t),(x,q,t),(x,r,t) \in \mathcal{D}$, for all $i \in N$, and for all $\lambda \in (0,1)$, we have $\varphi_i(x,p,t) \geq \varphi_i(x,q,t)$ if and only if $\varphi_i(x,\lambda p+(1-\lambda)r,t) \geq \varphi_i(x,\lambda q+(1-\lambda)r,t)$.

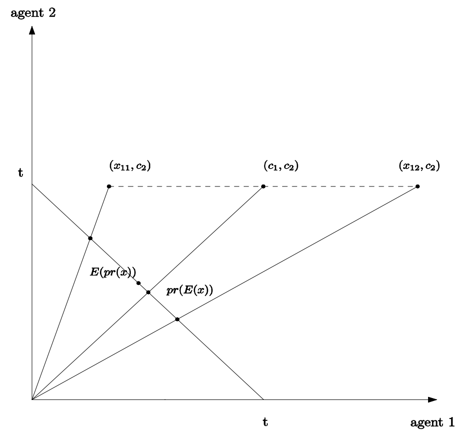

Figure 4: Mean-preserving spread makes agent 1 worse off when agent 2 has a deterministic claim.

Theorem 3 *Let $|N| \geq 3$. A rationing rule φ satisfies NARAI, AN, SYM, IND, CONT, and NAN if and only if φ is ex-post proportional rule.*

Proof. "If" part is obvious. We will prove the "only if" part. Let $(x, p, t) \in \mathcal{D}$. Let φ be a rationing rule satisfying NARAI, AN, SYM, IND, CONT, and NAN. By Theorem 1 and NAN, we have

$$\varphi_i(x, p, t) = \sum_{s \in S} [(W_s(x_N, p, t) x_{is})] \tag{7}$$

.

By CONT and IND we can invoke celebrated Expected Utility Characterization of von Neumann and Morgenstern (1947) and deduce that φ_i should be additively separable with respect to probabilities. That is, for all $x \in \mathbb{R}^{N \times S}$ for all $p \in \Delta^{|S|-1}$ for all $i \in N$ and for all $s \in S$ there exists $u_{is} : \mathbb{R}^{N \times S} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\varphi_i(x, p, t) = \sum_{s \in S} [p_s u_{is} (x, t)] \tag{8}$$

.

By (7) and (8) we deduce that for all $s \in S$ there exists $v_s : \mathbb{R}^S \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\varphi_i(x, p, t) = \sum_{s \in S} [p_s x_{is} v_s(x_N, t)] \tag{9}$$

Consider a degenerate lottery δ_s, that is, fix $s \in S$ and let $p_s = 1$.

So $\varphi_i(x, \delta_s, t) = x_{is} v_s(x_N, t)$.
Summing over $i \in N$, we get $\sum\limits_{i \in N} \varphi_i(x, \delta_s, t) = \sum\limits_{i \in N} [x_{is} v_s(x_N, t)] = v_s(x_N, t) x_{Ns} = t$. So $v_s(x_N, t) = \frac{t}{x_{Ns}}$. Hence we get the ex-post proportional rule.
$\varphi_i(x, p, t) = \sum\limits_{s \in S} [p_s x_{is} v_s(x_N, t)] = \sum\limits_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) t$. ■

4 Conclusion

We studied a rationing problem where the claims are state contingent. We introduce two extensions of the proportional rules in our framework – ex-ante and ex-post proportional rules. Applying the proportional rule to the expected claim gives the ex-ante proportional rule. Ex-post proportional rule is defined as the expectation of the shares given by the proportional rule for various states. To characterize these rules we propose two extensions of No Advantageous Reallocation introduced by Moulin(1985). The first extension, *NARAI*, requires that no group of agents benefits if transfers are allowed across individuals for each state whereas the second extension, *NARAS*, considers transfers across states. We characterize ex-ante proportional rule by *NARAI* and *NARAS* combined with Anonymity, Symmetry, Continuity, and No Award for Null. To characterize ex-post proportional rule, we introduce an Independence axiom similar to the one used in the Expected Utility Theory. This axiom says that by mixing two lotteries with a third one, the rationing rule remains unaffected by the choice of the third lottery. Replacing *NARAS* with the aforementioned Independence axiom gives the characterization of ex-post proportional rule.

This paper leads us to two particularly important questions to be considered in future research. The first question is to find axiomatic characterizations of the extensions of other important rules, such as, Uniform Gains, Uniform Losses, etc. It will also be interesting to extend our framework to situations like, (a) where the resource itself is state contingent, and (b) the individuals are taking subjective probabilities, that is, the probabilities for the states are not uniform across the individuals.

5 Appendix

Proposition 1 *Let $(x, p, t) \in \mathcal{D}$ be given. The difference between the shares of an individual given by ex-ante and ex-post proportional rules is given by the following.*

$$\overline{pr}_i(x,p,t) - \widetilde{pr}_i(x,p,t) = \frac{\sum\limits_{s\in S}\left[p_s x_{is}\left(1 - p_s - \sum\limits_{j\in N}\frac{x_{js}}{x_{is}}\sum\limits_{r\neq s}\frac{p_r x_{ir}}{x_{Nr}}\right)\right]}{\sum\limits_{s\in S}(p_s x_{Ns})}t$$

Proof. *Define $\alpha_{js} = \frac{x_{js}}{x_{is}}$ for all $j \in N$ and for all $s \in S$.*
So ex-ante proportional rule for agent i is given by

$$\overline{pr}_i(x,p,t) = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{Ns})}t = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})}t$$

And ex-post proportional rule for agent i is given by

$$\widetilde{pr}_i(x,p,t) = \sum_{s\in S}\left(p_s\frac{x_{is}}{x_{Ns}}\right)t = \sum_{s\in S}\frac{p_s}{\alpha_{Ns}}t$$

So the difference between ex-ante and ex-post proportional rule is

$$\overline{pr}_i(x,p,t) - \widetilde{pr}_i(x,p,t) = \frac{\sum\limits_{s\in S}(p_s x_{is})}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})}t - \sum_{s\in S}\frac{p_s}{\alpha_{Ns}}t =$$

$$= \frac{\sum\limits_{s\in S}(p_s x_{is})\prod\limits_{s\in S}\alpha_{Ns} - \sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})\prod\limits_{s\in S}\alpha_{Ns}\sum\limits_{s\in S}\frac{p_s}{\alpha_{Ns}}}{\prod\limits_{s\in S}\alpha_{Ns}\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})}t =$$

$$= \frac{\prod\limits_{s\in S}\alpha_{Ns}\left[\sum\limits_{s\in S}(p_s x_{is}) - \sum\limits_{s\in S}[p_s x_{is}\alpha_{Ns}]\sum\limits_{s\in S}\frac{p_s}{\alpha_{Ns}}\right]}{\prod\limits_{s\in S}\alpha_{Ns}\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})}t =$$

$$= \frac{\sum\limits_{s\in S}\left[\left(p_s - p_s^2\right)x_{is}\right] - \sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})\sum\limits_{r\neq s}\frac{p_r}{\alpha_{Nr}}}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})}t = \frac{\sum\limits_{s\in S}\left[p_s x_{is}\left(1 - p_s - \sum\limits_{j\in N}\alpha_{js}\sum\limits_{r\neq s}\frac{p_r}{\alpha_{Nr}}\right)\right]}{\sum\limits_{s\in S}(p_s x_{is}\alpha_{Ns})}t =$$

$$\frac{\sum\limits_{s\in S}\left[p_s x_{is}\left(1 - p_s - \sum\limits_{j\in N}\frac{x_{js}}{x_{is}}\sum\limits_{r\neq s}\frac{p_r x_{ir}}{x_{Nr}}\right)\right]}{\sum\limits_{s\in S}(p_s x_{Ns})}t. \quad \blacksquare$$

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