Auctions with Resale under State Uncertainty

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Abstract

In this paper, we study equilibria of second price auctions with resale when the value of the object is subject to future state uncertainty and ordering of bidder use values differ in each state. Equilibrium bids are higher than expected use values and increasing as bidder types move away from the median type. Due to this convexity, information conveyed by the bids is insufficient to identify the type of the losing bidder under incomplete information, creating further uncertainty for resale pricing.

Keywords: Auctions, Resale, Uncertainty

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1 Introduction

Most asset values are subject to some form of future state risk. In the case of unique and indivisible assets, depending on how uncertainty resolves, the agent with the highest value can vary. This suggests that there are scenarios, in which there is a likelihood of trade after the resolution of uncertainty. In this paper, we provide a two-period stylized model for optimal bidding behavior of agents, if the seller auctions the asset prior to the resolution of this uncertainty and there exists a possibility of post-auction trade.[1]

In our second price auction model, individuals have future state-contingent use values[2] and the ordering of use values across bidders changes in different states, creating a resale possibility for the winner of the auction.

The importance of resale market in auctions has been undermined since in the absence of informational linkages between auction and resale stages, equilibria of auctions with or without resale are identical (Haile (1999)). Haile (2003) has studied auctions with resale in an environment where bidders receive a signal about the future use value of the object which creates an incentive and possible efficiency gains from the resale market. Gupta and Le Brun (1999) analysed first price auctions with resale where bidder valuations are not identically distributed and first stage bids are announced prior to the resale stage. Hafalir and Krishna (2008) studied the equilibria of first and second price auctions with resale with asymmetric bidders which creates an incentive for an active resale market. Virag (2013) extends their analysis to the case with many bidders. Garratt and Troger (2006) introduced a speculator, a bidder with commonly known zero use value, and explored the equilibria of auctions with resale and symmetric independent private value bidders, where the existence of the speculator is the main motive which leads to an active resale market. In an analysis of treasury bill market, Bikhchandani and Huang (1989), models primary market as a

[1]This also implies that, in our environment, any ex-ante allocation can be ex-post inefficient, regardless of the auction mechanism used to allocate the asset.

[2]Throughout this paper, in line with Haile (1999), we adopt the term "use value" to define bidders' exogenous state-contingent values and the term "valuation" to denote endogenously determined ex-ante expected value of the asset to the bidders.

common value auction with the purpose of resale and investigate the role of information linkages between auction stage and the resale market on the optimal bidding behaviour. In this paper, we assume all parameters of the model are common knowledge for the bidders, except the other bidder's type. In this sense, our model can be seen as an immediate extension of Vickery's (1962) seminal work.

In these papers, the potentiality of resale is driven by the resolution of uncertainty of bidders on their own valuations. In our model, resolution of the state uncertainty creates resale opportunities. As a natural consequence, in the absence of resale, ex-post efficiency cannot be guaranteed under our assumptions.[3] Our results indicate that existence of this type of uncertainty has major implications on the bidding behavior.

We consider an environment with two possible future states, where two risk-neutral bidders share the same expected use value. In the absence of resale (post-auction trade), bidders would bid this expectation and the asset would be allocated probabilistically. Thus, main motivation behind common expected value assumption is to isolate the effects of state uncertainty. As a result, under this assumption, competitive bidding is driven only by potential resale profits and therefore resulting price in excess of expected use value can be regarded as option value of resale.

We consider two possible scenarios. First, we focus on the case where bidders know only their own types in the auction stage but types become common knowledge prior to the resale stage. We show that in a symmetric equilibrium, bidding strategies are non-monotone, strictly convex and attains a minimum at the median of the type distribution. Intuitively, when bidders move towards the extremes of the type distribution they will bid more aggressively to win the auction. Bidders on the extremes of the type distribution are those who have a large bias towards either of the states in terms of their use value and these bidders still have a high valuation on the object because of the resale opportunities. If the undesired state realizes, it is always possible to sell the object to the bidder with a higher valuation at that state and extract all surplus. Note that, this

[3] It is easy to show that this statement is false only if the model guarantees existence of a dominant bidder, i.e. a bidder who has the highest valuation in all possible states of future.

surplus (resale profit) is increasing in the difference between bidder use values in either state.

Second, we analyse the case where there is no revelation of types in any stage of the game, however the winning bidder can extract information about losing bidders type, because her bid is the price winning bidder pays to purchase the object. Since the bidding strategies are non-monotone, the winning bidder can not identify the exact type of the losing bidder even under truthful bidding. This, in addition to type and state uncertainty, creates identification risk in the resale stage under incomplete information. In other words, in equilibrium, after state uncertainty is resolved (at resale stage), due to non-monotonicity of the bidding function, winner of the first stage auction still has a risk of mispricing the object, which diminishes (in certain cases hinders) her capacity to extract all surplus. For this case, we construct a symmetric equilibrium, which, under some conditions, exhibits similar properties with the first case, such as non-monotone and convex bidding function and bids are increasing in distance to the mean.

The paper is organized as follows: Section 2 defines the environment and the model. Section 3 provides the equilibrium for the case where bidder types reveal before the resale stage. In Section 4, we provide incomplete information equilibrium where the winner of first stage auction only observes the bid of the losing bidder. Finally, Section 5 concludes.

2 Environment and Model

2.1 Environment

A unique and indivisible object is for sale. There are two risk-neutral bidders, $i \in I = \{1, 2\}$, who attach no sentimental value to the object. There are two time periods, $t = 0, 1$, where $t = 0$ is the auction stage and $t = 1$ is the resale stage. At the auction stage, a second price auction is held by a seller with a value zero and object is allocated to the highest bidder.[4]

At the resale stage there are two possible states of nature, $\mathcal{S} = \{u, d\}$, which realize with

[4] If bidders submit the same bid, object is allocated to a player with a probability of $\frac{1}{2}$.

a positive probability, $0 < P(s) < 1, s \in \mathcal{S}$. Bidders' use values depend on the realized state and they do not derive any utility from the object until the end of the resale stage. There is no discounting. We denote the use value of bidder i in state s as v_i^s. At the resale stage, first-stage winner makes a take-it-or-leave-it offer to the other bidder by setting monopoly price.

An immediate implication of the environment is regarding the efficiency of the allocation and the existence of a resale market.

Proposition 1 *There is always an ex-ante possibility of gains from resale stage trade in the absence of a bidder i such that $v_i^s \geq v_{-i}^s, \forall s \in \mathcal{S}$*[5].

Necessity of introducing a resale market and allocative efficiency is widely discussed in the auction literature.It is argued that, in the absence of information linkages between stages and under certain assumptions, auction mechanisms that yield optimal allocations exist without a resale market.[6] However, since the use values of individuals can vary over time in an uncertain environment, possibility of achieving the optimal allocation (regardless of the auction mechanism) requires the existence of a "dominant" type bidder, who has the highest use value in all states with certainty, to whom object should be allocated.

The proposition also implies that whenever there is a different bidder with the maximum use value in each state then the existence of a resale market always improves upon the allocation obtained by an auction without resale. This observation is one of the main motivations of this paper. Note that, Proposition (1) stems from the environment, implying that any auction mechanism which allocates the object in the first stage can not eliminate gains from resale.[7] When individuals have state-dependent use values, it is necessary to introduce a resale market, especially because one cannot guarantee the existence of a dominant bidder as defined in Proposition (1).

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[5]This result also holds for environments with multiple bidders or states

[6]See Haile (2003) and Zheng (2002) for a detailed discussion on the efficiency of auction mechanism and the possibility of resale.

[7]Unless the model can guarantee existence of a dominant bidder who, with certainty, will have higher valuation in all states.

2.2 Model

In this section, we present our model. Recall that $\mathcal{S} = \{u, d\}$, and let probabilities associated with up and down states be $P(s = u) = p$ and $P(s = d) = 1 - p$, $p \in (0, 1)$. These probabilities are common knowledge.

Expected use value of the object for each bidder is given by:

$$Ev_i = pv_i^u + (1-p)v_i^d, \ \forall i \in I.$$

We assume equality of expected use values:

Assumption 1. $v_i^s \geq 0, \forall s.$

Assumption 2. $Ev_i = v, \quad \forall i \in I$

Given Assumption 2, we normalize v_i^s, such that:

$$\theta_i = \frac{v_i^u}{v} \quad \gamma_i = \frac{v_i^d}{v}$$

which yields the following one-to-one correspondence between θ and γ values:

$$\gamma_i = \frac{1 - p\theta_i}{1 - p}, \ \forall i \in I \tag{1}$$

As a result, while each bidder i has a unique pair of (θ_i, γ_i), and θ_i is a sufficient statistic to uniquely identify the type of bidder i.

Assumption (2) enables us to observe the effects of future state uncertainty in the absence of differences in expected use values hence isolate the effects of uncertainty. Additionally, it eliminates the possibility of a dominant bidder discussed in Proposition (1) and guarantees the possibility of resale trade in the model.

Assumption 3 . Each θ_i is assumed to be identically and independently distributed according to a continuous, strictly increasing cumulative distribution $F(\cdot)$ with a continuous density $f(\cdot)$

over the support $[\underline{\theta}, \overline{\theta}]$ [8]. We also assume $F(\cdot)$ is twice continuously differentiable on $(\underline{\theta}, \overline{\theta})$.

Prior to the auction stage, each bidder independently draws her θ_i. These draws are privately known to the bidders.

Figure 1 illustrates the restriction on type space induced by Assumption 2 on $(\underline{\theta}, \overline{\theta})$, which also yields a negative linear relationship between state-contingent use values.

The alternating orderings of use values across different states drives the resale incentives in our model, which can occur under two general scenarios. Figure 2 represents a scenario where $\underline{\theta}$ is greater than one (i.e. market-wide risk); uncertainty affects bidders' use values in the same direction (relative to their common expected use value) but in different magnitudes. Whereas, Figure 3 demonstrates a scenario where $\underline{\theta}$ is less than one (i.e. industry-level risk); bidders obtain their highest possible use values in different states. Note that, in both scenarios, there is no dominant bidder.

In the following sections, we provide equilibria of the model under two different information structures: (i) the case where bidders know only their own types in the auction stage but types become common knowledge prior to resale stage. (ii) the case where there is no information revelation in any stage of the game. However winning bidder can extract information about losing bidder's type, because her bid is the price winning bidder pays to purchase the object.

We find the equilibria of the game where bidder types are revealed prior to resale stage in Section 3. The equilibrium characterization for the case with no information revelation at the resale stage is provided in Section 4. Section 5 concludes.

[8]This assumption necessitates the support $[\frac{1-p\overline{\theta}}{1-p}, \frac{1-p\underline{\theta}}{1-p}]$ for γ values. Also, given the non-negativity of use values,v_i^s, we have $\underline{\theta} \geq 0$ and $\overline{\theta} \leq \frac{1}{p}$.

2

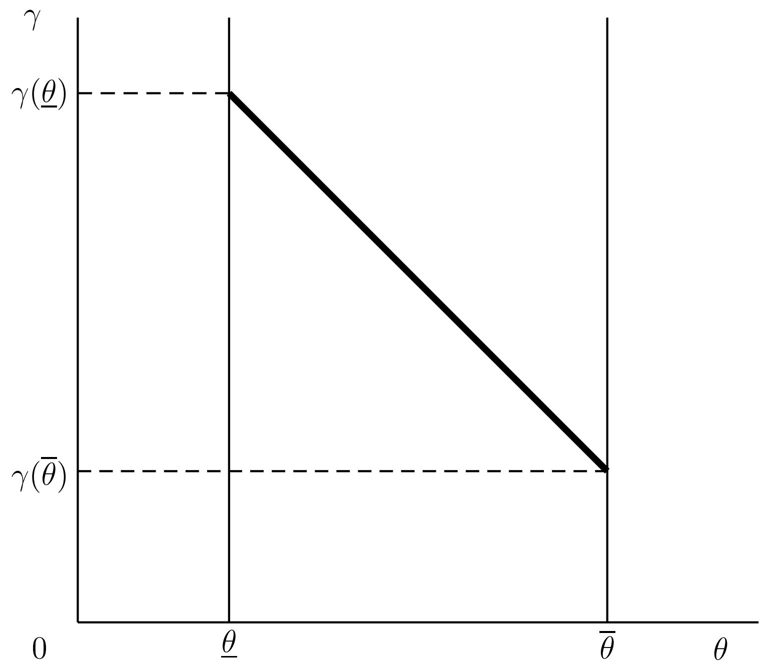

Figure 1: Restriction imposed on the type space by Assumptions 2 and 3

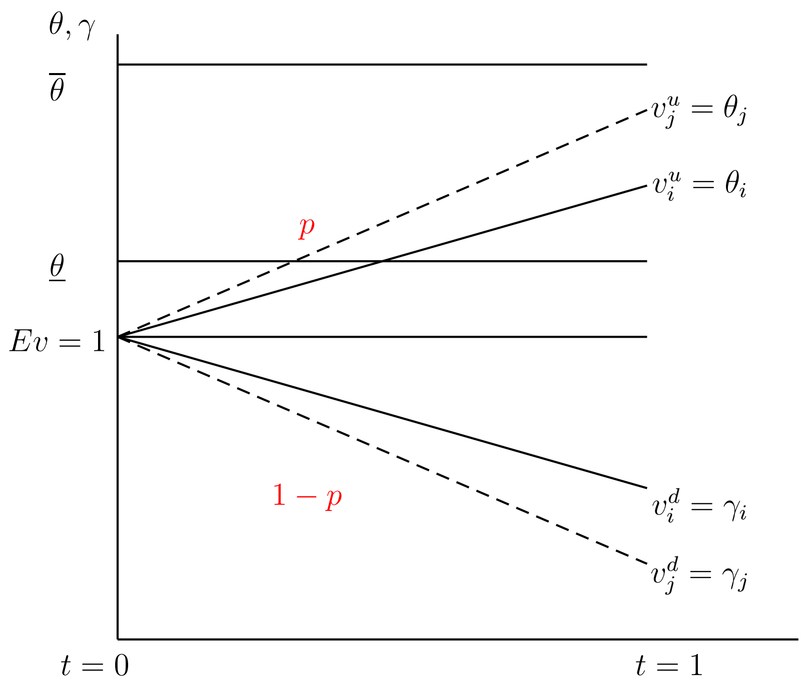

Figure 2: Auction stages: $\underline{\theta} > 1$

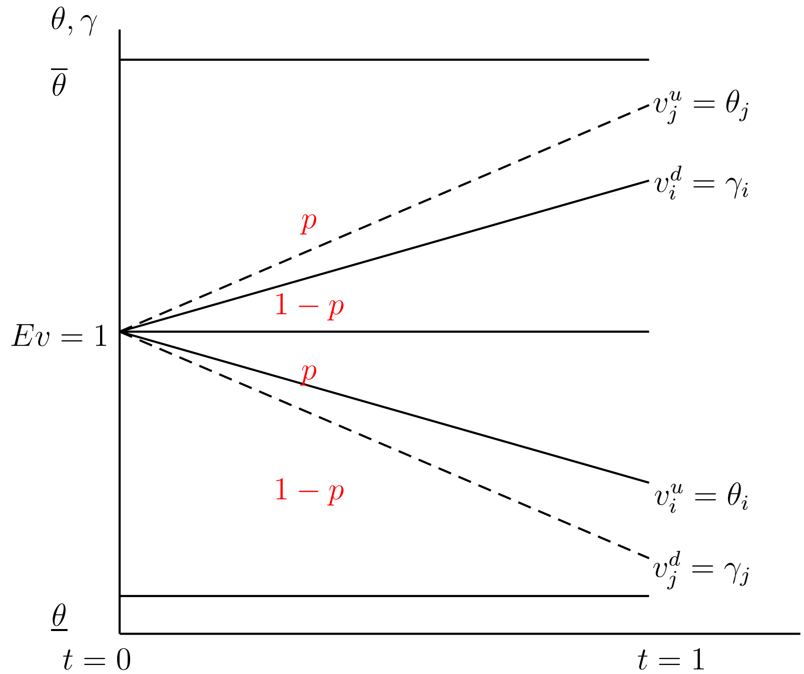

Figure 3: Auction stages: $\underline{\theta} < 1$

3 Complete Information at the Resale Stage

In this section, we consider that there is complete information at the resale stage and describe the possible outcomes of the game. Types are common knowledge at the resale stage , therefore we use backward induction to find equilibria.

At the resale stage, if the losing bidder has higher use value in the realized state, the winning bidder of auction stage sells the object to the losing bidder. Since all types are common knowledge, winner offers the object to the loser, setting the monopoly price which is exactly the use value of losing bidder. Loser accepts this offer in equilibrium and obtains zero terminal utility. If the winning bidder has the highest value in the realized state, she consumes the object obtaining her use value.

By Proposition (1) and Assumption (2), there is always an ex ante possibility of resale, depending on the relative value of the winner's θ and the realization of the state. Bidders will face uncertainty during the auction stage about whether they have higher or lower θ.

3.1 Auction Stage

In this section, we will find the equilibrium of the auction stage game, given the resale stage strageties described above. We characterize the equilibrium of the game using backward induction.

At time $t = 0$, prior to auction stage, θ_i and θ_j are drawn and private information, without loss of generality we assume $\theta_i > \theta_j$. At the resale stage, when the uncertainty realizes, types will reveal and become common knowledge. As a result both bidders face uncertainty about whether they are the low or high type (θ) bidder before they submit their bids and hold a belief about the other players' draw prior to the auction stage, conditional on her draw of θ. First, we construct the beliefs of the agents about the other agent's type conditional on her type. Recall that $\theta_i \sim F$ with support $[\underline{\theta}, \overline{\theta}]$. Let the probability of bidder i having maximum θ as $\mu_i = P(\theta_i = \tilde{\theta}_{max}|\theta_i)$ and having maximum γ (therefore minimum θ) as $\eta_i = P(\theta_i = \tilde{\theta}_{min}|\theta_i)$ [9]. Consequently, we have $1 - \mu_i = P(\theta_i < \tilde{\theta}_{max}|\theta_i)$ and $1 - \eta_i = P(\theta_i > \tilde{\theta}_{min}|\theta_i)$.[10]

Note that, as in Haile (2003), valuations of bidders are different than their expected use values. This is because bidders' valuations are endogenously determined and due to the resale opportunity and the valuation of bidders (if they win) are strictly greater than their expected use values, consequently their bids.

Bidders face uncertainty about their relative valuations in each state. Therefore, given their type, they form expectations about their state dependent valuations, i.e. $(\tilde{W}_i^d, \tilde{W}_i^u)$. Expected valuation of bidder i is the average of state valuations weighted by state probabilities reads:

$$V_i = pE(\tilde{W}_i^u) + (1-p)E(\tilde{W}_i^d) \tag{2}$$

where state dependent valuations depend on the probability of having the highest θ or γ, which can be written as:

[9] $\sim$ detones a random variable.

[10] Note that $\theta_i < \theta_{max}$ if and only if $\theta_i \neq \theta_{max}$ and $\theta_i > \theta_{min}$ if and only if $\theta_i \neq \theta_{min}$

$$V_i = p\{\mu_i E[\theta_{max}|\theta_i = \theta_{max}] + (1-\mu_i)E[\theta_{max}|\theta_i < \theta_{max}]\}$$
$$+(1-p)\{\eta_i E[\gamma_{max}|\gamma_i = \gamma_{max}] + (1-\eta_i)E[\gamma_{max}|\gamma_i < \gamma_{max}]\}$$

Noting that the winning bidder will not offer the object for resale if she has the maximum value in that state, or equivalently she offers for resale at a price equal to her value, therefore the above expression simplifies to:

$$V_i = p\big\{\mu_i\theta_i + (1-\mu_i)E[\theta_{max}|\theta_i < \theta_{max}]\big\} + (1-p)\big\{\eta_i\gamma_i + (1-\eta_i)E[\gamma_{max}|\gamma_i < \gamma_{max}]\big\} \quad (3)$$

Writing the associated conditional probabilities and substituting γ we obtain[11]:

$$V_i = 1 + p\bigg\{\theta_i\big[F(\theta_i) - (1-F(\theta_i))\big] - \int_{\underline{\theta}}^{\theta_i} xd\big[1-(1-F(x))\big] + \int_{\theta_i}^{\overline{\theta}} xdF(x)\bigg\} \quad (4)$$

which reduces to:

$$V_i = 1 + p\bigg\{\overline{\theta} - \underline{\theta} - \big[\int_{\underline{\theta}}^{\theta_i}(1-F(x))dx + \int_{\theta_i}^{\overline{\theta}} F(x)dx\big]\bigg\} > 1 \quad (5)$$

The expected valuation V_i can be regarded as the private value in a standard second price auction and similarly in equilibrium both bidders bid their private value, which is the expected valuation for our environment. However, while bidding functions are monotone in the expected valuation V_i, they are not monotone in θ_i. In the following section we formalize the observation on this novel non-monotonicity of the bidding function.

We focus on symmetric Bayesian equilibria of the game. Observe that, by construction, optimal strategies of the players depend on the value of θ_i, because of the link between θ and γ induced

[11] Detailed derivation is provided in the Appendix.

by common expected use values. and the parameters of the game, we assume both agents will use symmetric strategies, i.e. $\beta_i = \beta_j = \beta$ where $\beta : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}^+$.

Equation (7) states the expected valuations of bidders conditional on the realization of θ. Proposition (3) states the familiar result that in a second-price auction bidders bid their private values.

Proposition 2 *In a second price sealed bid auction followed by a resale stage under complete information with monopoly pricing,*

$\beta(\theta_i) = V_i, \quad for \;\; i = 1, 2$

is an equilibrium bidding strategy.

Proof. *See Appendix.*

The price that the seller expects to get in the case of incomplete information at the auction stage is given by:

$$EP = \int_{\underline{\theta}}^{\overline{\theta}} \left[1 + p[\overline{\theta} - \underline{\theta} + \int_{\underline{\theta}}^{y} (1 - F(x))dx + \int_{y}^{\overline{\theta}} F(x)dx] \right] dH(y) \tag{6}$$

where $H(y) = 1 - (1 - F(y))^2$ is the cumulative distribution function of the second order statistic.

3.2 Behaviour of the Expected Valuations

The most important implication of the state uncertainty introduced in our environment is the non-monotone behaviour of the endogenous expected valuations, that is the expected valuations of the bidders are such that two different values of θ may yield to the same expected valuation, which is stated in the following proposition.

Proposition 3 *Let κ be the median of $f(\cdot)$. Then β is continuous, strictly convex, and has a minimum at $\theta = \kappa$.*[12]

[12] This proposition is also true for the case with finitely many bidders.

Proof. *See Appendix.*

Proposition 4 *If $f(\cdot)$ is symmetric, then β is symmetric around the mean of $f(\cdot)$, $E(\theta) = \kappa = \frac{\underline{\theta}+\overline{\theta}}{2}$.* ***Proof.*** *See Appendix.*

Proposition (4) implies that the behaviour of the expected valuations depends critically on the median of the type distribution, i.e the expected valuations decrease when a bidder's type gets closer to the median and gets larger when bidder's types move away from the median. When bidders move towards the extremes of the type distribution they will bid more aggressively to win the auction. The rationale behind this observation is that for two bidders who are on the extremes of the type distribution, their expected valuation behave similarly. Bidders on the extremes of the type distribution are the ones who have a large bias towards one of the states of the world in terms of the use value, i.e. use value is very high in one state and very low in the other. Observe that such bidders still have a high valuation on the object because of the resale opportunities. If the undesired state realizes, it is always possible to sell the object to the bidder with a higher valuation at that state and extract the surplus.

Assuming symmetry in the type distribution allows us to obtain a sharper characterization on the expected valuations of the bidders such that they are symmetric around the expected valuation of the mean type, that is bidders valuations depends solely on the distance of their type from the mean type and expected valuations of two bidders equidistant to the mean are the same. Therefore the expected valuations can be written solely as function of distance to the mean. We elaborate this issue further when we analyse the case of incomplete information at the resale stage.

4 Incomplete Information at the Resale Stage

In this section, we consider the case where types do not reveal in any stage of the game and the winning bidder infer the private value of the losing bidder through her bid.

There are two bidders. Winner of the first stage auction observes the losing bid, that is actually the price she pays, and if there are gains from trade she sells the object through a take-it-or leave-it offer at the resale stage. She conditions the price at the resale stage on the information inferred from the losing bid to maximize her expected payoff.

A strategy for bidder i consists of a bidding strategy, $\beta_i : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}^+$ and, for each state, a pricing strategy, π_i^s , if she wins the auction and an acceptance strategy, A_i^s, if she loses the auction. Pricing strategies, $\pi_i^s : [b_j, \theta_i, \gamma_i] \to \mathbb{R}^+$, characterize the optimal asking price in the resale for each state. We assume that losing bidder will accept any resale price greater or equal to her value, i.e. the set of prices she will accept are: $A_i^u = \{\pi_j^u : \pi_j^u \leq \theta_i\}$ at state u and $A_i^d = \{\pi_j^d : \pi_j^d \leq \gamma_i\}$ at state d.

Definition 1 *A perfect Bayesian equilibrium of the game consists of bidding strategies β_i, pricing strategies π_i^s, acceptance strategies A_i^s and belief functions μ_i for $i = 1, 2$ and $s = u, d$ such that: (1) if bidder i loses the auction, the set of resale prices she will accept are given by A_i^u and A_i^d (2) if bidder i wins the auction then π_i^s is optimal given μ_i and the acceptance strategy of bidder j (3) for each θ_i, β_i is optimal given β_j, π_i and π_j (4) beliefs are generated from F and β_i using Bayes rule whenever possible.*

For the rest of the analysis, we assume that $f(\theta)$ is symmetric. We define $z(\theta) =\mid \theta - E(\theta) \mid$ as the distance of bidder's type from the mean of the type distribution. Thus, $z^{-1}(\theta)$ has at most two elements, denoted as $\theta_{i,l}$ and $\theta_{i,h}$, where $\theta_{i,l} = \min\{z_i^{-1}(\theta)\}$ and $\theta_{i,h} = \max\{z_i^{-1}(\theta)\}$. Note that $\gamma = \frac{1-p\theta}{1-p}$ by construction, therefore, there are two possible types corresponding to the same z value, $[\theta_{i,h}, \gamma_{i,l}]$ and $[\theta_{i,l}, \gamma_{i,h}]$[13]. We denote the cumulative probability distribution of z as $G(z) : [0, \frac{\underline{\theta}+\overline{\theta}}{2}] \to [0, 1]$ with the associated density function $g(z)$.

We focus on symmetric perfect Bayesian equilibria of the game. Suppose, bidder j follows a bidding strategy $\beta(\theta_j)$ which is (i) continuous and (ii) symmetric around $E(\theta)$ and (iii) strictly increasing in z. We have $\beta(\theta_{j,h}) = \beta(\theta_{j,l})$ by supposition and define $\tilde{\beta}(z) : [0, \frac{\underline{\theta}+\overline{\theta}}{2}] \to \mathbb{R}^+$ as follows:

[13]We adapt this notation two emphasize relative positions of γ values given corresponding θ values.

$$\tilde{\beta}(z_j) = \beta(\theta_{j,h}) = \beta(\theta_{j,l}) \tag{7}$$

4.1 Resale Stage and Pricing Strategy

We suppose that bidder j follows β, and bidder i wins the auction and pays the bid of j, $b_j = \tilde{\beta}(z_j)$. $\tilde{\beta}$ is strictly increasing in z, bidder i infers that $z_i > \hat{z}_j$ where $\hat{z}_j = \tilde{\beta}^{-1}(b_j)$. Bidder i formulates her pricing strategies, π_i^u and π_i^d based on the information she will infer from b_j. By equation (7), there are two values of θ associated with $\hat{z}_j$:

$$\beta^{-1}(b_j) = \left\{\theta : \tilde{\beta}(z_j) = b_j\right\} = \left\{\hat{\theta}_{j,h}, \hat{\theta}_{j,l}\right\} \tag{8}$$

where type vector of bidder j is either $[\hat{\theta}_{j,h}, \hat{\gamma}_{j,l}]$ -high type- or $[\hat{\theta}_{j,l}, \hat{\gamma}_{j,h}]$ -low type-. Bidder i attaches a probability 1/2 to each possible outcome.

First, we characterize the pricing decision of a high type bidder i with $[\theta_{i,h}, \gamma_{i,l}]$. Bidder i acts as a monopolist at the resale stage and maximize her expected payoff given the inference from b_j. In the up state, a high-type bidder i, with a relatively higher use value, will not sell the object[14]. However, in the down state, bidder i has a relatively lower use value ($\gamma_{i,l} < \hat{\gamma_{j,l}} < \hat{\gamma_{j,h}}$), thus resale is profitable. The optimal price depends on the relative positions of z_i and $\hat{z}_j = \tilde{\beta}^{-1}(b_j)$. The following claim characterizes the optimal pricing strategy for the bidder with $[\theta_{i,h}, \gamma_{i,l}]$ (high type bidder).

Claim 2 ***(Pricing strategy for the high type)*** *For $s = d$, given the acceptance strategies, A_j^u and A_j^d, if bidder i wins the first stage auction with type $[\theta_{i,h}, \gamma_{i,l}]$ sets a resale price $\pi_i^d = \hat{\gamma}_{i,l}$ if $z_i > 3\hat{z_j}$ and sets $\pi_i^d = \hat{\gamma}_{i,h}$ if $z_i < 3\hat{z_j}$ [15]. For $s = u$, there is no resale.*

[14] Equivalently, she sets a price equal to her use value and is not able to sell given the acceptance strategy of bidder j.

[15] An alternative pricing strategy is to set a resale price equal to the average of the values of possible low and high type values of bidder j. However, it is not optimal since if bidder j is low type, bidder i can not sell the object and if bidder j is of high type, it reduces resale profit. The same argument holds for any price between $\gamma_{i,l}$ and $\gamma_{i,h}$.

Table 1. Possible Outcomes of Pricing Game under Incomplete Information

Winning bidder sets price	Possible Types of Losing Bidder l	Possible Types of Losing Bidder h
l	Correct pricing: Resale occurs, winning bidder extracts all surplus	Mispricing: Resale occurs, losing bidder also makes positive profit
h	Mispricing: Resale does not occur, winning bidder consumes the object	Correct pricing: Resale occurs, winning bidder extracts all surplus

Proof: See Appendix.

Similarly, Claim (3) provides the pricing strategy of bidder i with type $[\theta_{i,l}, \gamma_{i,h}]$ (low type bidder).

Claim 3 ***(Pricing strategy for the low type)*** *For $s = d$, given A_j^u and A_j^d, if bidder i wins the first stage auction with type $[\theta_{i,l}, \gamma_{i,h}]$ wins the auction, there is no resale. For $s = u$, bidder i sets a price $\pi_i^u = \hat{\theta}_{j,h}$ if $z_i < 3\hat{z}_j$ and sets $\pi_i^u = \hat{\theta}_{j,l}$ if $z_i > 3\hat{z_j}$.*

Proof: See Appendix.

Claims (2) and (3) describe the optimal pricing strategy of bidder i if she wins the auction. If bidder i loses the auction and bidder j bids according to β, given the pricing strategies of bidder j, then there is a possibility to gain a positive payoff due to mispricing of winning bidder, j, at the resale stage. Table 1 provides possible outcomes of the pricing game.

A losing bidder i obtains a positive gain from resale, if bidder j sets a price equal to the low-type inferred from $\hat{z}_i$ where bidder i is actually a high-type bidder, i.e. there is an opportunity for bidder i due to mispricing of bidder j. The following claim formalizes the argument.

Claim 4 ***(Gain from mispricing)*** *Given the pricing strategies, the payoff of a losing bidder i with type $\theta_{i,h}$ is equal to $\theta_{i,h} - \theta_{i,l}$, if $s = u$ and $z_i < \frac{z_j}{3}$, zero otherwise. Similarly, the payoff for the losing bidder of type $\theta_{i,l}$ is $\gamma_{i,h} - \gamma_{i,l}$, if $s = d$ and $z_i < \frac{z_j}{3}$, zero otherwise.*

Proof: See Appendix.

4.2 Auction Stage

In this section, we derive the valuation of bidder i at the auction stage given the pricing strategies at the resale stage. We formulate these valuations for bidder i of high and low types, i.e. $[\theta_{i,h}, \gamma_{i,l}]$ and $[\theta_{i,l}, \gamma_{i,h}]$.

First, we focus on the valuation of a high-type bidder i if she wins the first stage auction. By supposition, bidder j follows β, thus the probability of winning for bidder i is equal to $G(z_i)$. In the up state, a high type bidder i will not sell the object then the valuation of a winning bidder i is equal to:

$$W(\theta_{i,h}, u) = G(z_i)\theta_{i,h} \tag{9}$$

In the down state, bidder i sells the good at the resale stage given the pricing strategy as in Claim (2). If $\frac{z_i}{3} < z < z_i$ - with probability $G(z_i) - G(\frac{z_i}{3})$- winning bidder sets resale price equal to the expected value of high type bidder j, $\hat{\gamma}_{j,h}$. With probability 1/2, bidder j is of low type, there is no resale and bidder i obtains her use value and with probability 1/2, bidder j is of high type and bidder i sells the object. If $z < \frac{z_i}{3}$ - with probability $G(\frac{z_i}{3})$ - winning bidder sets resale price equal to the expected value of low type bidder j and sells the object. Hence, valuation of a winning bidder i is formulated as:

$$W(\theta_{i,h}, d) = \left[G(z_i) - G(\frac{z_i}{3})\right]\left(\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\hat{\gamma}_{j,h} \mid \frac{z_i}{3} < z < z_i]\right) + G(\frac{z_i}{3})E[\hat{\gamma}_{j,l} \mid z < \frac{z_i}{3}] \tag{10}$$

Similarly, the valuations of the low type bidder i, i.e. $[\theta_{i,l}, \gamma_{i,h}]$ in the up and down states is given by:

$$W(\theta_{i,l}, u) = \left[G(z_i) - G(\frac{z_i}{3})\right]\left(\frac{1}{2}\theta_{i,l} + \frac{1}{2}E[\hat{\theta}_{j,h} \mid \frac{z_i}{3} < z < z_i]\right) + G(\frac{z_i}{3})E[\hat{\theta}_{j,l} \mid z < \frac{z_i}{3}] \tag{11}$$

$$W(\theta_{i,l}, d) = G(z_i)\gamma_{i,h} \tag{12}$$

Ex-ante expected valuation of bidder i, conditional on winning the first stage auction, is equal to:

$$V_W = pW(\theta_{i,t}, u) + (1-p)W(\theta_{i,t}, d) \ \ t \in \{l, h\} \tag{13}$$

If bidder i loses the first stage auction, she may obtain the object at the resale stage. As claim (5) establishes, high type bidder i obtains a positive payoff only if bidder j sets a the low resale price $\hat{\theta}_{i,l}$ at the state which happens whenever $z_i > \frac{\hat{z}_j}{3}$. The valuation of a losing bidder i is given by:

$$V_L(\theta_{i,h}) = p\frac{1}{2}\left[1 - G(3z_i)\right]\left(\theta_{i,h} - \hat{\theta}_{j,l}\right) \tag{14}$$

where $\hat{\theta}_{j,l}$ is the inference of bidder j on the type of bidder i given b_i. Similarly, for the losing bidder i with the low type, i.e. $[\theta_{i,l}, \gamma_{i,h}]$ we obtain the expected payoff as:

$$V_L(\theta_{i,l}) = (1-p)\frac{1}{2}\left[1 - G(3z_i)\right]\left(\gamma_{i,h} - \hat{\gamma}_{j,l}\right) \tag{15}$$

Under incomplete information there is a possibility of ex-ante gains from losing at the first stage auction, thus we define the total valuation as the sum of the expected payoffs from winning and losing as:

$$V = V_W + V_L \tag{16}$$

Expected valuation depends on the type of bidder i and we denote the valuation of high type bidder i, i.e. $[\theta_{i,h}, \gamma_{i,l}]$ as $V^H(\theta_{i,h})$ which is equal to:

$$V^H(\theta_{i,h}) = G(z_i)\big[pW(\theta_{i,h}, u) + (1-p)W(\theta_{i,h}, d)\big] + EL(\theta_{i,h}) \tag{17}$$

First term in the expression gives the expected payoff of bidder i if she wins the first stage auction depending on the realized state , the second term, $EL(\theta_{i,h})$, is the expected pay-off obtained from resale in the case bidder i loses the first stage auction. Bidder i loses the first stage auction with probability $1 - G(z_i)$ and her expected pay-off is given by $(1/2)[1 - G(3z_i)](\theta_{i,h} - \hat{\theta}_{i,l}) + (G(3z_i) - G(z_i)) \times 0$ and the probability of the event is already embedded in the valuation.

Similarly, for the low-type i.e. $[\theta_{i,l}, \gamma_{i,h}]$, the expected valuation is given by:

$$V^L(\theta_{i,l}) = G(z_i)\big[pW(\theta_{i,l}, u) + (1-p)W(\theta_{i,l}, d)\big] + EL(\theta_{i,l}) \tag{18}$$

Equations (19) and (20) give the expected valuations of bidder i depending on being low or high type, facing a bidder j who follows a bidding strategy β which is continuous, strictly increasing and symmetric around the mean type. The following proposition establishes that bidder i with a low type has the same expected valuation as the bidder i of high type, thus expected valuations are symmetric i.e. bidder i's expected valuation can be written as a function of z, distance to the mean type.

Proposition 5 *Suppose that bidder j follows $\beta(.)$ and resale prices $[\pi_i^u, \pi_i^d]$ and $[\pi_j^u, \pi_j^d]$, given the acceptance strategies $[A_i(u),\ A_i(d)]$ and $[A_j(u)$,$A_j(d)]$, are formulated as described in claims 2, 3 and 4, then we have $V^H(\theta_{i,h}) = V^L(\theta_{i,l}) = V(z_i)$ where $z_i = \theta_{i,h} - E(\theta) = E(\theta) - \theta_{i,l}$.*

Proof. *See Appendix.*

Rearranging the terms, expected valuation conditional on winning can be written as a function of z for bidder i:

$$W(z_i) = 1 + pz_i + \frac{p}{G(z_i)}\Big[\int_0^{\frac{z_i}{3}} G(x)dx - \frac{1}{2}\int_{\frac{z_i}{3}}^{z_i} G(x)dx\Big] \tag{19}$$

and $V_W = G(z_i)W(z_i)$. Equation (21) is important in two aspects. First, pay-off from winning the first stage auction is always greater than one, which is greater than the ex-ante expected use value of the object. Formally, we have $V_W(\theta) > 1 = v$, $\forall\theta \in [\underline{\theta}, \overline{\theta}]$, hence there is an incentive for overbidding for both type of bidders similar to the case with complete information. Second, incomplete information at the resale stage pricing creates an information cost due to the mispricing, last term in equation (21), which is always negative.

Both bidder maximize the ex-ante expected payoff from the game when constructing their bidding strategies given the pricing and bidding strategy of the other bidder. Given that bidder j follows the strategy described above, the expected payoff of bidder i reads:

$$V(z_i) = V_W(z_i) + V_L(z_i) \tag{20}$$

where $V_L(z_i) = p[1 - G(3z_i)]z_i$.

The following proposition establishes the properties of the expected valuation from winning is continuous and strictly increasing in z under a sufficiency condition imposed on the cumulative distribution of z.

Proposition 6 *Expected valuation from winning $V_W(z)$ is continuous and strictly increasing in z.*

Proof. *See Appendix.*

Proposition (6) states that the result for the complete information at the resale stage with expected payoffs increasing in the distance to the mean-type holds true for the case with incomplete information and expected valuations are still higher than the expected value which is set equal to one for both bidders. Expected valuation of bidder is monotone in z even though it is non-monotone in θ domain which helps us identify bidders in terms of their distance to the mean type.

We want to construct an equilibrium bidding strategy for the second price auction which is held at the first stage. The information revealed prior to the resale stage is not sufficient to uniquely identify the type of the losing bidder. This creates identification risk for the winning bidder and reduces her expected pay-off. On the other hand, both bidders always have an incentive to overbid because the expected pay-off is still higher than the expected use value, which creates the primary trade-off for the bidders when formulating their optimal bidding strategies. Note that we have both incomplete information and uncertainty in this environment which results in a novel problem where standard arguments from the auction theory fail to hold where all bidders bid their expected valuations as in the case with complete information. Both bidders have an incentive to overbid and there are ex-ante profitable deviations from the strategy prescribing $\beta(z) = V(z)$. [16]

Let $\phi(z_i) = [1 - G(3z_i)]z_i$. We propose a linear bidding rule given by $\beta(z) = 1 + p\frac{\phi(\lambda)}{2} + pz_i$ for $z \in [0, \overline{\theta} - \kappa]$ constitutes an equilibrium.

Proposition 7 *In the second price auction, pricing strategies $\pi_i = [\pi_i^u, \pi_i^d]$ described in claims 2, 3 and 4, acceptance strategies $[A_i(u), A_i(d)]$ for $i = 1, 2$ and the bidding strategies described as*

$$\beta(z_i) = 1 + p\frac{\phi(\lambda)}{2} + pz_i$$

constitutes an equilibrium where $\lambda = \underset{z}{\operatorname{argmax}}\, \phi(z)$, if and only if $G(\cdot)$ satisfies the following condition:

$$3\int_0^{\frac{z_i}{3}} G(x)dx + \int_0^{z_i} G(x)dx \geq G(z_i)\phi(\lambda) - 2\phi(z_i), \qquad \forall z_i < 2\phi(\lambda) \tag{21}$$

Proof: See Appendix.

Proposition (7) establishes a linear equilibrium bidding function under a sufficiency condition on $G(\cdot)$ which guarantees that all types of bidders have an incentive to participate in the auction. In our environment, besides the strong overbidding incentive, there is a possible gain from losing the first stage auction, which can be thought as speculation. Bidders with low z values, i.e. close

[16] See appendix for a proof that $\beta(z) = V(z)$ is not equilibrium bidding strategy.

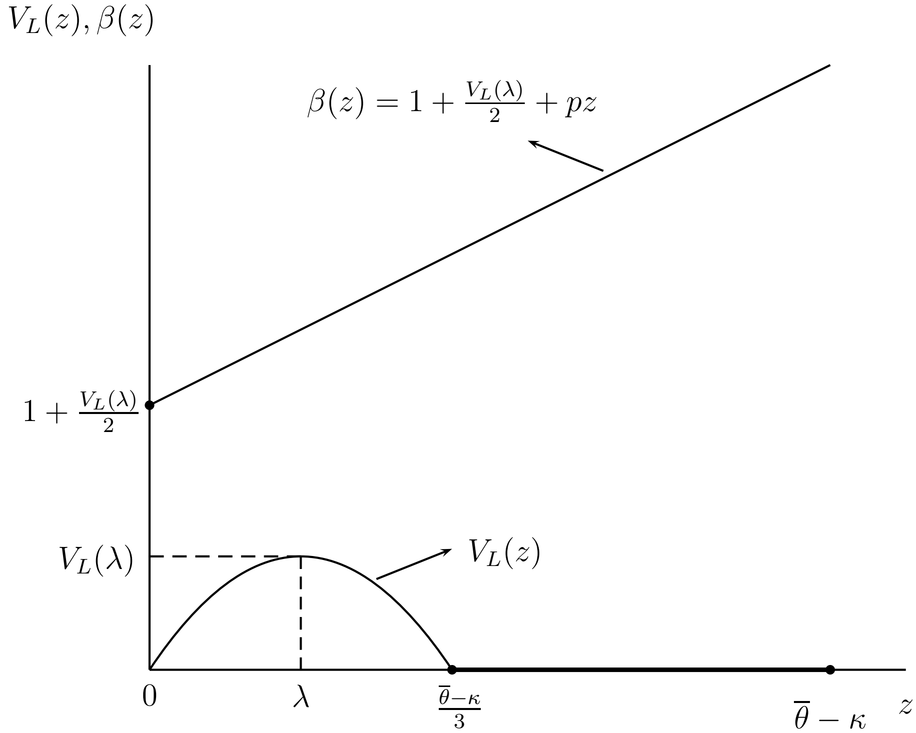

Figure 4: Equilibrium Bidding Function

to the mean type with low expected valuation, have an incentive for signalling a different type to the winning bidder through the bid at the first stage auction and gain a higher pay-off at the resale stage. Proposed bidding strategy requires a significant amount of overbidding by bidders which discourages low type bidders from deviating from the prescribed strategy. The condition states that the probability distribution has thin tails which supports the incentive for overbidding by providing a sufficiently high probability of winning to the bidder with relatively higher z values.

Note the the proposed strategy posits a similar behaviour of the equilibrium bidding function as in the case with complete information at the resale stage. Under incomplete information, we obtain a similar behaviour of the equilibrium bidding function as the one with complete information at the resale stage where bids are increasing as bidders move away from the mean type.

5 Conclusion

When unique and indivisible assets with values subject to future state uncertainty are auctioned, optimal bidding behaviour or optimal pricing will critically depend on the resolution of the uncertainty, since depending on the realized state, the agent who values the asset most is subject to change. In this paper, we provide a two-period stylized model for optimal bidding behaviour of agents, if the seller of such an asset auctions it prior to the resolution of this uncertainty, and given that there exists a possibility of ex-post trade.

We present a model for second price auction with resale where bidders have state-contingent use values with an altering order in different states, creating a resale possibility for the winner of the auction. Therefore, in our environment, any ex-ante allocation can be ex-post inefficient depending on the realization of uncertainty. In a two-player two state environment with risk-neutral bidders, we characterize equilibria for complete and incomplete information at the resale stage. We assume that state -contingent values are non-negative and bidders have the same expected value for the object. The latter assumption aims at isolating the effects of uncertainty and induce an altering order of use values under different state realizations. Under complete information at the resale stage, we show that in a symmetric equilibrium, optimal bidding strategies are non-monotone, strictly convex and attains a minimum at the median of the type distribution.

Fundamental market based asset pricing models either assume divisible assets or perfectly liquid markets and prices of these financial assets depend on the uncertainty prevails in the financial markets. However, not all financial assets exhibit divisibility and liquidity such as privately held companies, intellectual property rights, toxic assets, failed banks etc. which renders market based asset pricing models inapplicable. Natural venue of analysis in such environments is to rely on the auction theory to be able to price these assets. Potential buyers of these assets form subjective valuations which are usually based on possible future scenarios, such as state of macroeconomic conditions or industry level uncertainties.

In this paper, we study second price sealed bid auctions with resale where the value of the asset is subject to state uncertainty. We assume that both bidders face the aggregate uncertainty however they have idiosyncratic exposures to it. In our environment, bidder with the highest use value of the asset alternates between states and both have the same expected valuation which allow us to isolate the effect of uncertainty. We show that, in general, there is an ex-ante possibility of gains from resale trade. Thus banning resale may not be socially optimal and also decreases seller's revenue in our model.

We model a second price sealed bid auction game where two risk-neutral bidders, who have the same expected use value but different exposures to risk, bid for an object. We solve the model for two different information revelation scenarios. Equilibrium bidding function is convex and non-monotone in either state-dependant use value, where minimum is achieved at the median of the support of distribution. If distribution is symmetric, bids are increasing not in types but in distance of types to the mean of the support. In other words, bidders bid more aggresively, as their bias towards a either state increase.

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6 Appendix

6.1 Expected Valuation under Incomplete Information in Auction Stage

Suppose there are n bidders. Then cumulative distribution function of the highest order statistic is $F_1(\cdot) = F^n(\cdot)$ and for the lowest order statistic is $F_n(\cdot) = 1 - [1 - F(\cdot)]^n$. From the perspective of bidder i, probability of bidder i being the highest type and bidder i being the smallest type are given by $F^{n-1}(\theta_i)$ and $1 - [1 - F(\theta_i)]^{n-1}$ respectively.

Recall that $\theta_i \sim F : [\underline{\theta}, \overline{\theta}] \to [0, 1]$, $\gamma_i = \frac{1-p\theta_i}{1-p}$. and the probability of having the maximum value of θ and γ for bidder i are given by $\mu_i = F^{n-1}(\theta_i)$ and $\eta_i = (1 - F(\theta_i))^{n-1}$.

We want to show that the expression in Equation (8) equals to the one in Equation (10). First note that:

$$E[\theta_{max}|\theta_i < \theta_{max}] = \frac{\int_{\theta_i}^{\theta_{max}} xdF^{n-1}(x)}{1 - F^{n-1}(\theta_i)}$$

$$E[\gamma_{max}|\gamma_i < \gamma_{max}] = E[\frac{1 - p\theta_{min}}{1 - p}|\theta_i > \theta_{min}] = \frac{1}{1-p}(1 - p\frac{\int_{\underline{\theta}}^{\theta_i} xd[1 - (1 - F(x)^{n-1}]}{1 - (1 - F(\theta_i))^{n-1}})$$

Substituting these expressions in Equation (4), we obtain:

$$\begin{aligned} V_i &= p\Big\{F^{n-1}(\theta_i).\theta_i + (1 - F^{n-1}(\theta_i))\Big[\frac{\int_{\theta_i}^{\overline{\theta}} xdF^{n-1}(x)}{1 - F^{n-1}(\theta_i)}\Big]\Big\} \\ &\quad +(1-p)\Big\{(1 - F(\theta_i))^{n-1}\gamma_i + [1 - (1 - F(\theta_i))^{n-1}]\Big[\frac{1}{1-p}(1 - p\frac{\int_{\underline{\theta}}^{\theta_i} xd[1 - (1 - F(x)^{n-1}]}{1 - (1 - F(\theta_i))^{n-1}})\Big]\Big\} \end{aligned}$$

Then it is straightforward to show that:

$$V_i = 1 + p\{\overline{\theta} - \theta_i + \int_{\underline{\theta}}^{\theta_i} [1 - (1 - F(x))^{n-1}]dx - \int_{\theta_i}^{\theta_{max}} F^{n-1}(x)dx\}$$

6.2 Proof of Proposition 3

Ex-ante expected pay-off of bidder i in a second price auction is given by the difference between her expected valuation described above and the second highest bid. Assume that the bid of agent j is b_j, then the expected pay-off EP_i is given by:

$$EP_i = V_i - b_j \tag{22}$$

Without loss of generality, consider bidder i. By bidding V_i, if $V_i > b_j$ she obtains a pay-off equal to this difference and zero otherwise.

Case 1: Suppose bidder i bids $\underline{b} < V_i$. Then if $V_i > \underline{b} > b_j$, bidder i still wins and obtains an expected pay-off of $V_i > b_j$ and if $V_i > b_j > \bar{b}$ she will obtain an expected pay-off zero. Therefore bidding less than the expected valuation can not be optimal because it can never increase her expected pay-off and in some cases bidder i will be worse off in terms of expected pay-off.

Case 2: Suppose bidder i bids $\overline{b} > V_i$. If $\overline{b} > V_i > b_j$ then she is indifferent, however if $\overline{b} > b_j > V_i$ overbidding reduces the expected payoff. Thus overbidding is also weakly dominated by bidding the expected value.

Hence $b_i = V_i$ is the weakly dominant equilibrium bidding strategy of a second-price auction followed by an optimal resale auction. Similar argument holds for bidder j.

6.3 Proof of Proposition 4

We provide a proof for the case with $n \geq 2$ bidders.

By definition, $F(\kappa) = \frac{1}{2}$ and by proposition (5), equilibrium bidding function is $\beta(\theta_i) = V_i$.

β is continuous, since it is sum of continuous functions. The derivative with respect to θ

obtains:

$$\begin{aligned}
\frac{\partial \beta}{\partial \theta_i} &= p\{F^{n-1}(\theta_i) - (1-F(\theta_i))^{n-1} + \theta_i[(n-1)F^{n-2}(\theta_i)f(\theta_i) - (n-1)(1-F(\theta_i))^{n-2}(-f(\theta_i))] \\
&\quad -\theta_i(n-1)F^{n-2}(\theta_i)f(\theta_i) - \theta_i(n-1)[1-F(\theta_i)]^{n-2}f(\theta_i)\} \\
\Rightarrow \frac{\partial \beta}{\partial \theta_i} &= p\{F^{n-1}(\theta_i) - (1-F(\theta_i))^{n-1}\}
\end{aligned}$$

We want to show that $\theta = \kappa$ is the unique minimum of $\beta(\cdot)$. Setting the derivative equal to zero we get:

$\frac{\partial \beta}{\partial \theta_i} = 0 \Leftrightarrow F^{n-1}(\theta_i) = (1-F(\theta_i))^{n-1}$. Since $F(\theta_i) \geq 0$,

which implies,

$\frac{\partial \beta}{\partial \theta_i} = 0 \Leftrightarrow F(\theta_i) = 1 - F(\theta_i) \Leftrightarrow F(\theta_i) = \dfrac{1}{2}$

The second derivative is equal to

$$\frac{\partial^2 \beta}{\partial \theta_i^2} = p\{(n-1)F^{n-2}(\theta_i)f(\theta_i) - (n-1)(1-F(\theta_i))^{n-2}(-f(\theta_i))\} > 0$$

Hence β is strictly convex and κ is the unique minimum, which also implies $\frac{\partial \beta}{\partial \theta_i} < 0$ for $\theta_i < \kappa$ and $\frac{\partial \beta}{\partial \theta_i} > 0$ for $\theta_i > \kappa$. ■

6.4 Proof of Proposition 6

For $n = 2$, we have $\beta(\theta_i) = 1 + p\{\overline{\theta} - \underline{\theta} - [\int_{\underline{\theta}}^{\theta_i}(1-F(x))dx + \int_{\theta_i}^{\overline{\theta}} F(x)dx]\}$ and symmetry of $f(\cdot)$ implies that $E(\theta) = \kappa = \frac{\overline{\theta}+\underline{\theta}}{2}$.

We need to show $\beta(\kappa-\alpha)=\beta(\kappa+\alpha),\forall\alpha$.

$\beta(\kappa-\alpha)=\beta(\kappa+\alpha)\Leftrightarrow \int_{\underline{\theta}}^{\kappa-\alpha}(1-F(x))dx+\int_{\kappa-\alpha}^{\overline{\theta}}F(x)dx=\int_{\underline{\theta}}^{\kappa+\alpha}(1-F(x))dx+\int_{\kappa+\alpha}^{\overline{\theta}}F(x)dx$

$\Leftrightarrow -\alpha-\int_{\underline{\theta}}^{\kappa-\alpha}F(x)dx+\int_{\kappa-\alpha}^{\overline{\theta}}F(x)dx=\alpha-\int_{\underline{\theta}}^{\kappa+\alpha}F(x)dx+\int_{\kappa+\alpha}^{\overline{\theta}}F(x)dx$

$\Leftrightarrow \alpha=\int_{\kappa-\alpha}^{\kappa+\alpha}F(x)dx \quad (*)$

We now prove (*) holds. Let $g(x)=F(x+\kappa)-\frac{1}{2}$. By symmetry of $f(\cdot)$, g is an odd function. Then,

$\int_{-\alpha}^{\alpha}g(x)dx=\int_{-\alpha}^{\alpha}(F(x+\kappa)-\frac{1}{2})dx=\int_{\kappa-\alpha}^{\kappa+\alpha}(F(x)-\frac{1}{2})=0$

$\Leftrightarrow \int_{\kappa-\alpha}^{\kappa+\alpha}F(x)dx=\int_{\kappa-\alpha}^{\kappa+\alpha}\frac{1}{2}dx=\alpha$ ■

6.5 Proof of Claim 2

Proof: We consider bidder i of type $[\theta_{i,h},\gamma_{i,l}]$. Observing b_j bidder i infers that bidder j is of type $[\hat{\theta}_{j,h},\hat{\gamma}_{j,l}]$ with probability 1/2 and of type $[\hat{\theta}_{j,l},\hat{\gamma}_{j,h}]$ with probability 1/2.

First consider the up state, i.e. $S=u$. Bidder i wins at the auction stage and infers $z_i>z_j=\tilde{\beta}^{-1}(b_j)$, hence bidder j is either of type $\hat{\theta}_{j,h}$ or $\hat{\theta}_{j,l}$. Bidder i is of type $\hat{\theta}_{i,h}$ and since $z_i>\hat{z}_j$ we have $\theta_{i,h}>\hat{\theta}_{j,h}>\hat{\theta}_{j,l}$ hence there is no resale.[17]

In the down state, i.e. $S=d$, bidder i of type $\gamma_{i,l}$ infers that bidder j is of type $\hat{\gamma}_{j,h}$ or $\hat{\gamma}_{j,l}$ with probability 1/2. Since $z_i>\hat{z}_j$, we have $\gamma_{i,l}<\hat{\gamma}_{j,l}<\hat{\gamma}_{j,h}$, hence there is room for resale. She chooses the resale price maximizes the expected profit.

1. If she sets $\pi_i^d=\hat{\gamma}_{j,l}$ expected profit is equal to $\hat{\gamma}_{i,l}$ since bidder j will buy the object for each possible type.

[17] Note that we may consider this situation as if bidder i sets a price equal to his value at the good state and if bidder j follows β, the price offer is outside of her acceptance set.

2. If she sets a price $\pi_i^d = \hat{\gamma}_{j,h}$ then the expected profit is given by $\frac{1}{2}\gamma_{i,l} + \frac{1}{2}\hat{\gamma}_{j,h}$ since only bidder j of type $\hat{\gamma}_{j,h}$ will buy the object.

Note that setting any price $\pi_i^d \in (\hat{\gamma}_{j,l}, \hat{\gamma}_{j,h})$ is not optimal since bidder i can always increase her payoff by setting a higher price. The optimal price depends on the relative positions of z_i and $\hat{z}_j$.

Bidder i sets $\pi_i^d = \hat{\gamma}_{j,l}$ when $\hat{\gamma}_{j,l} > \frac{1}{2}\gamma_{i,l} + \frac{1}{2}\hat{\gamma}_{j,h}$. Equivalently, whenever $\gamma_{i,l} < 2\hat{\gamma}_{j,l} - \hat{\gamma}_{j,h}$ which translates to $z_i > 3\hat{z}_j$ in the z domain. She sets $\pi_i^d = \hat{\gamma}_{j,h}$ otherwise.

6.6 Proof of Claim 3

Proof: We consider bidder i of type $[\theta_{i,l}, \gamma_{i,h}]$. Observing b_j bidder i infers that bidder j is of type $[\hat{\theta}_{j,h}, \hat{\gamma}_{j,l}]$ with probability 1/2 and of type $[\hat{\theta}_{j,l}, \hat{\gamma}_{j,h}]$ with probability 1/2.

First consider the up state, i.e. $S = u$. Bidder i of type $\theta_{i,l}$ infers that bidder j is of type $\hat{\theta}_{j,h}$ or $\hat{\theta}_{j,l}$ with probability 1/2. Since $z_i > \hat{z}_j$, we have $\theta_{i,l} < \hat{\theta}_{j,l} < \hat{\theta}_{j,h}$, hence there is room for resale. She chooses the resale price maximizes the expected profit.

1. If she sets $\pi_i^u = \hat{\theta}_{j,l}$ expected profit is equal to $\hat{\theta}_{i,l}$ since bidder j will buy the object for each possible type.

2. If she sets a price $\pi_i^u = \hat{\theta}_{j,h}$ then the expected profit is given by $\frac{1}{2}\gamma_{i,l} + \frac{1}{2}\hat{\gamma}_{j,h}$ since only bidder j of type $\hat{\theta}_{j,h}$ will buy the object.

In the down state, i.e. $S = d$, bidder i is of type $\hat{\gamma}_{i,h}$ and since $z_i > \hat{z}_j$ we have $\gamma_{i,h} > \hat{\gamma}_{j,h} > \hat{\gamma}_{j,l}$ hence there is no resale.

Bidder i sets $\pi_i^u = \hat{\theta}_{j,l}$ when $\hat{\theta}_{j,l} > \frac{1}{2}\theta_{j,l} + \frac{1}{2}\hat{\theta}_{j,l}$ or equivalently $\theta_{i,l} < 2\hat{\theta}_{j,l} - \hat{\theta}_{j,h}$ which implies $z_i > 3\hat{z}_j$ and sets a price $\pi_i^u = \hat{\theta}_{j,h}$ otherwise.

6.7 Proof of Claim 4

Proof: Consider bidder i of type $[\theta_{i,h}, \gamma_{i,l}]$ loses at the auction stage. Payoff of bidder i depends on the pricing strategy of bidder j which is set according to the inference given the bid of i. Bidder j infers that bidder i is of type $[\hat{\theta}_{i,h}, \hat{\theta}_{i,l}]$ with probability 1/2 and of type $[\hat{\theta}_{i,l}, \hat{\gamma}_{i,h}]$ with probability 1/2. Bidder j's strategy is the same as in claims (3) and (4).

Losing bidder i only infers that $z_i < z_j$ and constructs expectations on the possible resale prices of bidder j. Suppose that bidder i bids according to β. First consider $S = u$.

1. If bidder j is of high type with $\theta_{j,h}$ there is no resale since $z_i < z_j$ implies $\theta_{i,h} < \theta_{j,h}$ which happens with probability 1/2 from bidder i's perspective.

2. If bidder j is low type with $\theta_{j,l}$ there is room for resale. If bidder j sets the high price i.e. $\pi_j^u = \hat{\theta}_{i,h}$ then bidder i obtains zero profit from resale. If bidder j sets $\pi_j^u = \hat{\theta}_{i,l}$ then bidder i gets a positive payoff which happens whenever $z_i < z_j/3$ with probability $1 - G(3z_i)$.

If $S = d$, bidder i with type $\gamma_{i,l}$ either faces a resale price which is outside her acceptance region or pays her value under truthful bidding where there are no profits in the case of losing the auction stage.

For the bidder i of type $[\theta_{i,l}, \gamma_{i,h}]$ the problem is symmetric.

6.8 Proof of Proposition 7

Fix z_i. There are two types of bidder i corresponding z_i, $[\theta_{i,h}, \gamma_{i,l}]$ for the high type and $[\theta_{i,l}, \gamma_{i,h}]$ for the low type. Note that θ uniquely identifies bidder's type and $\gamma_{i,l} = \frac{1-p\theta_{i,h}}{1-p}$, $\gamma_{i,h} = \frac{1-p\theta_{i,l}}{1-p}$. By symmetry of $f(\theta)$, we have $\theta_{i,h} + \theta_{i,l} = 2\kappa$.

Let $A = G(z_i)$, $B = G(z_i) - G(\frac{z_i}{3})$, $C = G(\frac{z_i}{3})$ and $D = 1 - G(3z_i)$. Note that $A = B + C$.

Rewriting equations (22) and (23) we obtain:

$$V^H(\theta_{i,h}) = pA\theta_{i,h} + (1-p)\Big[B\Big(\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\gamma_{j,h} \mid \frac{z_i}{3} < z < z_i]\Big) + CE[\gamma_{j,l} \mid z < \frac{z_i}{3}]\Big] + \frac{D}{2}p(\theta_{i,h} - \theta_{i,l}) \tag{23}$$

$$V^L(\theta_{i,l}) = (1-p)A\gamma_{i,h} + p\Big[B\Big(\frac{1}{2}\theta_{i,l} + \frac{1}{2}E[\theta_{j,h} \mid \frac{z_i}{3} < z < z_i]\Big) + CE[\theta_{j,l} \mid z < \frac{z_i}{3}]\Big] + \frac{D}{2}(1-p)(\gamma_{i,h} - \gamma_{i,l}) \tag{24}$$

We want to show that $V^H(\theta_{i,h}) - V^L(\theta_{i,l}) = 0$.

$\gamma_{i,l} = \frac{1-p\theta_{i,h}}{1-p}$, $\gamma_{i,h} = \frac{1-p\theta_{i,l}}{1-p}$ and $\theta_{i,h} + \theta_{i,l} = 2\kappa$ implies that

$$pA\theta_{i,h} - (1-p)A\gamma_{i,h} = A\big[p2\kappa - 1\big] \quad (*)$$

and

$$\begin{aligned}(1-p)\Big[B\big(\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\gamma_{j,h}|\frac{z_i}{3} < z < z_i]\big) + CE[\gamma_{i,l}|z < \frac{z_i}{3}]\Big] \\ -p\Big[B\big(\frac{1}{2}\theta_{i,l} + E[\theta_{j,h}|\frac{z_i}{3} < z < z_i]\big) + CE[\theta_{j,l}|z < \frac{z_i}{3}]\Big] \\ = (B+C)[1 - p2\kappa] \quad (**)\end{aligned}$$

Given $A = B + C$, the sum of (*) and (**) is equal to zero. The equality of $\frac{1}{2}pD(\theta_{i,h} - \theta_{i,l}) - \frac{1}{2}(1-p)D(\gamma_{i,h} - \gamma_{i,l}) = 0$ simply follows from $\gamma_{i,l} = \frac{1-p\theta_{i,h}}{1-p}$ and $\gamma_{i,h} = \frac{1-p\theta_{i,l}}{1-p}$.

6.9 Proof of Proposition 6

Continuity of $V_W(z)$ directly follows from the continuity of the type distribution. Given the symmetry of $V_W(\theta)$, we focus only on the high type bidder. Expected valuation is :

$$V_W(\theta_{i,h}) = G(z_i)\big[pW(\theta_{i,h},u) + (1-p)W(\theta_{i,h},d)\big] \tag{25}$$

Using $\gamma_{i,l} = \frac{1-p\theta_{i,h}}{1-p}$ and $\gamma_{i,h} = \frac{1-p\theta_{i,l}}{1-p}$, and substituting $\theta_{i,h} = z_i + \kappa$ and $\theta_{i,l} = \kappa - z_i$, we rewrite (29) as:

$$V_W(z_i) = G(z_i) + p\left[\frac{z_i}{2}\big(G(z_i) + G(\frac{z_i}{3})\big) + \frac{1}{2}\int_{\frac{z_i}{3}}^{z_i} xg(x)dx - \int_0^{\frac{z_i}{3}} xg(x)dx\right]$$

Applying integration by parts we obtain:

$$V_W(z_i) = G(z_i) + p\left[z_iG(z_i) + \int_0^{\frac{z_i}{3}} G(x)dx - \frac{1}{2}\int_{\frac{z_i}{3}}^{z_i} G(x)dx\right] \tag{26}$$

Taking the derivative with respect to z_i we obtain:

$$\frac{\partial V_W}{\partial z} = g(z_i) + p\left[\frac{1}{2}G(z_i) + z_ig(z_i) + \frac{1}{2}G(\frac{z_i}{3})\right] > 0, \qquad \forall z_i > 0.$$

6.10 Proof of Proposition 7

Proposed strategy prescribes that $\tilde{\beta}(z) = 1 + V_L(\frac{\lambda}{2}) + pz_i, \forall z$. We first show that this strategy yields a positive expected payoff for all z. The expected payment of a winning bidder i, if bidder j follows $\tilde{\beta}(z)$ is given by:

$$EP(z_i) = G(z_i)E(\tilde{\beta}(z)|z < z_i) \tag{27}$$

The payoff of bidder i obtained as:

$$V(z_i) = G(z_i)\big[1 + pz_i + p\frac{1}{G(z_i)}[\frac{3}{2}\int_0^{\frac{z_i}{3}} G(x)dx - \frac{1}{2}\int_0^{z_i} G(x)dx]\big] + pz_i(1 - G(3z_i)) \tag{28}$$

We need to show that $V(z_i) \geq EP(z_i)$ for all z_i, i.e. there is an ex-ante incentive to participate in the first stage auction for any bidder.

This inequality can be written as:

$$G(z_i) + p\big[z_iG(z_i) + \frac{3}{2}\int_0^{\frac{z_i}{3}} G(x)dx - \frac{1}{2}\int_0^{z_i} G(x)dx\big] + pz_i(1 - G(3z_i) \geq G(z_i)E(1 + \frac{V_L(\lambda)}{2} + pz|z < z_i)$$
$$z_iG(z_i) + \frac{3}{2}\int_0^{\frac{z_i}{3}} G(x)dx - \frac{1}{2}\int_0^{z_i} G(x)dx + z_i(1 - G(3z_i) \geq G(z_i)\frac{\lambda(1 - G(3\lambda))}{2} + \int_0^{z_i} xg(x)dx$$
$$\frac{3}{2}\int_0^{\frac{z_i}{3}} G(x)dx + \frac{1}{2}\int_0^{z_i} G(x)dx \geq G(z_i)\frac{\lambda(1 - G(3\lambda))}{2} - z_i[1 - G(3z_i)](*)$$

(*) is the necessary and sufficient condition for participation of all θ_i to the auction. Letting $\phi(z_i) = [1 - G(3z_i)]z_i$, it reduces to:

$$3\int_0^{\frac{z_i}{3}} G(x)dx + \int_0^{z_i} G(x)dx \geq G(z_i)\phi(\lambda) - 2\phi(z_i), \qquad \forall z_i < 2\phi(\lambda). \tag{29}$$

Now, we need to show that $\tilde{\beta}(z) = 1 + \frac{V_L(\lambda)}{2} + pz$ is a best response by bidder i whenever she believes that bidder j bids according to $\tilde{\beta}(z)$. Define $V(z_i'; z_i)$ as the valuation of bidder i if she deviates from $\tilde{\beta}(z)$ and bids as if her type is z_i'.

We need to show that for all $z_i' \neq z_i$,

$$V(z_i) - G(z_i)E[\beta(z)|z < z_i] \geq V(z_i'; z_i) - G(z_i')E[\beta(z)|z < z_i'] \tag{30}$$

or equivalently;

$$V(z_i'; z_i) - V(z_i) \leq G(z_i')E[\beta(z)|z < z_i] - G(z_i)E[\beta(z)|z < z_i'](**) \tag{31}$$

Define the expected gain from deviation as $D(z_i'; z_i) = V(z_i'; z_i) - V(z_i)$, which consists of two components: the gain if i wins the first stage auction, $D_W(z_i'; z_i)$, and the gain if i loses the first stage auction, $D_L(z_i'; z_i)$, such that:

$$D(z_i'; z_i) = D_W(z_i'; z_i) + D_L(z_i'; z_i). \tag{32}$$

Given z_i, w.l.o.g., consider the high type bidder with $\theta_{i,h}$. Suppose that she is mimicking bidder with $z_i' \neq z_i$ in the auction stage.

$V(z_i', z_i)$ can also be decomposed to:

$$V(z_i', z_i) = V_W(z_i', z_i) + V_L(z_i', z_i) \tag{33}$$

Case 1: (Overbidding) $z_i' > z_i$

In the up state, if she wins, she is no longer the highest type with certainty. There exists $\theta_{j,h}$, such that $\theta_{i,h} < \theta_{j,h} < \theta_{i,h}'$. Thus, in the resale stage, due to deviation, a window for trade arises. Ex-ante up state value from winning can be written as:

$$G(z_i)\theta_{i,h} + \big(G(z_i') - G(z_i)\big)\left(\frac{1}{2}\theta_{i,h} + \frac{1}{2}E[\theta_{j,h} \mid z_i < z < z_i']\right) \tag{34}$$

In the down state, the set of bidders who will accept a resale offer expands, ex-ante. Therefore, in addition to his no deviation profits, there is a likelihood of additional profit because of the increase in the potential set of buyers at the resale stage.Thus, ex-ante down state payoff reads:

$$\begin{aligned}\big[G(z_i) - G(\frac{z_i}{3})\big]\left(\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\gamma_{j,h} \mid \frac{z_i}{3} < z < z_i]\right) + G\big(\frac{z_i}{3}\big)E[\gamma_{j,l} \mid z < \frac{z_i}{3}] \\ + \big(G(z_i') - G(z_i)\big)\big[\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\gamma_{j,h} \mid z_i < z < z_i']\end{aligned}$$

Combining the valuations in the up and down states, we obtain ex-ante valuation of winning from deviation to $z_i' > z_i$ as:

$$V_W(z_i', z_i) = p\Big[G(z_i)\theta_{i,h} + \big(G(z_i') - G(z_i)\big)\big(\frac{1}{2}\theta_{i,h} + \frac{1}{2}E[\theta^{j,h} \mid z_i < z < z_i']\Big]$$
$$+(1-p)\Big[[G(z_i) - G(\frac{z_i}{3})]\Big(\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\gamma_{j,h} \mid \frac{z_i}{3} < z < z_i]\Big) + G\big(\frac{z_i}{3}\big)E[\gamma_{j,l} \mid z < \frac{z_i}{3}]$$
$$+\big(G(z_i') - G(z_i)\big)\big(\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E[\gamma_{j,h} \mid z_i < z < z_i']\Big]$$

Then, $D_W(z_i', z_i) = V_W(z_i', z_i) - V_W(z_i)$ is given as:

$$D_W(z_i', z_i) = \big(G(z_i') - G(z_i)\big)\Big[\frac{1}{2} + \frac{1}{2}E\big[\theta^{j,h} + \gamma_{j,h} \mid z_i < z < z_i'\big]\Big] \tag{35}$$

Using the relation $\theta = \frac{1-p\gamma}{1-p}$ we write:

$$D_W(z_i', z_i) = \big(G(z_i') - G(z_i)\big)\Big[1 + pE\big[z \mid z_i < z < z_i'\big] \tag{36}$$

Case 2: (Underbidding) $z_i' < z_i$:

In this case, there are two subcases: (i) aggressive underbidding with $z_j < z_i' < \frac{z_i}{3}$ and (ii) moderate underbidding with$\frac{z_i}{3} < z_i' < z_i$.

If i wins with underbidding, $(z_j < z_i')$, in the up state, she is the highest type with certainty in both subcases,. Thus, her ex-ante valuation in the up state is $G(z_i')\theta_{i,h}$.

In the down state, the pricing rule does not change.

Case 2.1: (Aggressive Underbidding)

If $z_j < z_i' < \frac{z_i}{3}$, then she always sets the low price given the pricing strategy. Then, the ex-ante valuation is:

$$G(z_i')E\big[\gamma_{j,l} \mid z < z_i\big] \tag{37}$$

Then, $V_W(z_i', z_i) = pG(z_i')\theta_{i,h} + (1-p)G(z_i')E\big[\gamma_{j,l} \mid z < z_i\big]$.

$$D_w(z_i', z_i) = pG(z_i')\theta_{i,h} + (1-p)G(z_i')E\big[\gamma_{j,l} \mid z < z_i\big] - V_W(z_i) \tag{38}$$

Case 2.2: (Moderate Underbidding)

If $\frac{z_i}{3} < z_i' < z_i$, then the pricing rule leads to the ex ante valuation:

$$\Big[G(\frac{z_i)}{3}E\big[\gamma_{j,l} \mid z < \frac{z_i}{3}\big] + \big(G(z_i') - G(\frac{z_i}{3})\big)\big[\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E\big[\gamma_{j,h} \mid \frac{z_i}{3} < z < z_i'\big] \tag{39}$$

The expected valuation from deviation is given by:

$$V_W(z_i', z_i) = pG(z_i')\theta_{i,h} + (1-p)\Bigg(G(\frac{z_i}{3})E\big[\gamma_{j,l} \mid z < \frac{z_i}{3}\big] + \big(G(z_i') - G(\frac{z_i}{3})\big)\big[\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E\big[\gamma_{j,h} \mid \frac{z_i}{3} < z < z_i'\big] \tag{40}$$

and expected valuation for no deviation case is:

$$V_W(z_i) = pG(z_i)\theta_{i,h} + (1-p)\Big[(G(\frac{z_i}{3})E\big[\gamma_{j,l} \mid z < \frac{z_i}{3}\big] + \big(G(z_i) - G(\frac{z_i}{3})\big)\big[\frac{1}{2}\gamma_{i,l} + \frac{1}{2}E\big[\gamma_{j,h} \mid \frac{z_i}{3} < z < z_i\big] \tag{41}$$

Thus,

$$\begin{aligned} V_W(z_i', z_i) - V_W(z_i) = p\theta_{i,h}\big(G(z_i') - G(z_i)\big) + (1-p)\Bigg(&(G(z_i') - G(z_i))\frac{1}{2}\gamma_{i,l} \\ &+ \frac{1}{2}\big[\int_{\frac{z_i}{3}}^{z_i'} \gamma_{j,h} dG(z) - \int_{\frac{z_i}{3}}^{z_i} \gamma_{j,h} dG(z)\big]\Bigg) \end{aligned}$$

Denote $K = \int_{\frac{z_i}{3}}^{z_i'} \gamma_{j,h} dG(z) - \int_{\frac{z_i}{3}}^{z_i} \gamma_{j,h} dG(z)$ then $K = \int_{z_i}^{z_i'} \gamma_{j,h} dG(z) = E\big[\gamma_{j,h} \mid z_i < z < z_i'\big]\big(G(z_i') - G(z_i)\big)$.

Hence, profit from deviation can be written as:

$$
\begin{aligned}
V_W(z_i', z_i) - V_W(z_i) &= \big(G(z_i') - G(z_i)\big)\Big[p\theta_{i,h} + (1-p)\frac{1}{2}\gamma_{i,l} + \frac{1-p}{2}E\big[\gamma_{j,h} \mid z_i < z < z_i'\big] \\
&= \big(G(z_i') - G(z_i)\big)\Big[\frac{1}{2}p\theta_{i,h} + \frac{1}{2}p\theta_{i,h} + \frac{1}{2}(1-p)\gamma_{i,l} + \frac{1-p}{2}E\big[\gamma_{j,h} \mid z_i < z < z_i'\big] \\
&= \big(G(z_i') - G(z_i)\big)\Big[\frac{1}{2} + \frac{1}{2}p\theta_{i,h} + \frac{1-p}{2}E\big[\gamma_{j,h} \mid z_i < z < z_i'\big] \\
&= \big(G(z_i') - G(z_i)\big)\Big[\frac{1}{2} + \frac{1}{2}p(\kappa + z_i) + \frac{1-p}{2}E\big[\frac{1-p(\kappa - z)}{1-p} \mid z_i < z < z_i'\big] \\
&= \big(G(z_i') - G(z_i)\big)\Big[1 + \frac{pz_i}{2} + \frac{p}{2}E\big[z \mid z_i < z < z_i'\big]\Big].
\end{aligned}
$$

Deviation payoffs associated with over and underbidding can be summarized as:

$$
D_W(z_i'; z_i) = \begin{cases} [G(z_i') - G(z_i)][1 + pE[z|z_i < z < z_i']], & z_i' > z_i \\ [G(z_i') - G(z_i)][1 + \frac{p}{2}z_i + \frac{p}{2}E[z|z_i < z < z_i']], & \frac{z_i}{3} < z_i' < z_i \\ pG(z_i')\theta_{i,h} + (1-p)G(z_i')E\big[\gamma_{j,l} \mid z < z_i\big] - V_W(z_i), & z_i' < \frac{z_i}{3} \end{cases}
$$

$$
D_L(z_i'; z_i) = \frac{p}{2}[(1 - G(3z_i')(z_i' + z_i) - (1 - G(3z_i))2z_i] \leq \frac{V_L(\lambda)}{2} \tag{42}
$$

We can rewrite change in expected payment as:

$$
G(z_i')E[\tilde{\beta}(z)|z < z_i'] - G(z_i)E[\tilde{\beta}(z)|z < z_i] = [G(z_i') - G(z_i)](1 + \frac{V_L(\lambda)}{2} + pE[z|z_i < z < z_i']) \tag{43}
$$

For $z_i' > z_i$, (**) can be written as:

$$
1 + pE[z|z_i < z < z_i'] + D_L(z_i'; z_i) \leq 1 + \frac{V_L(\lambda)}{2} + pE[z|z_i < z < z_i'], \tag{44}
$$

is true, since $D_L(z_i'; z_i) \leq \frac{V_L(\lambda)}{2}$.

For $\frac{z_i}{3} < z_i' < z_i$, (**) implies:

$$1 + \frac{p}{2}z_i + \frac{p}{2}E[z|z_i < z < z_i']] + D_L(z_i'; z_i) \leq 1 + \frac{V_L(\lambda)}{2} + pE[z|z_i < z < z_i'], \tag{45}$$

holds since $\frac{p}{2}z_i + \frac{p}{2}E[z|z_i < z < z_i'] < pE[z|z_i < z < z_i']$ and $D_L(z_i'; z_i) \leq \frac{V_L(\lambda)}{2}$.

For the case $z_i' < \frac{z_i}{3}$, it can be shown that

$$pG(z_i')\theta_{i,h} + (1-p)G(z_i')E\left[\gamma_{j,l} \mid z < z_i\right] - V_W(z_i) < [G(z_i') - G(z_i)][1 + \frac{p}{2}z_i + \frac{p}{2}E[z|z_i < z < z_i']]$$

thus, there is no profitable deviation, which completes the proof.